B8.3: Mathematical Modelling of Financial Derivatives

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1 Binomial trees

Assume that someone gives you a tricked coin. How would you work out the probability of the coin landing in tails, $\mathbb{P}[\text{Tails}] = ?$ Perhaps the simplest way is to run an experiment, i.e., Monte Carlo, where one tosses the coin n times and then records the number of successes S (i.e., Tails) and that will give you a good proxy for $\mathbb{P}[\text{Tails}] = S/n$.

Such an approach is a bit more difficult in finance. We cannot perform experiments (or at least perform them in a cost-effective way). What we can do, instead, is to propose a model for the underlying movements of the stock price; simulate price paths and then calculate the price, say, of a European option. But what is a good model? A tractable one? One that conforms with empirical evidence? Or a combination of both?

Below we discuss in detail the binomial model of option pricing. Although simple, the binomial model is not just of pedagogical use because one can show that, if generalised to multiple periods, the prices it yields converge to those given by the

Black–Scholes model, where the stochastic dynamics of the stock is driven by Brownian motion. (Brownian motion shall be discussed in detail later on.)

Before proceeding, we note the distinction between different approaches used to price instruments. *Pricing by arbitrage* is based on the idea that if two instruments have exactly the same payoff in all states of nature then their price must be the same. *Equilibrium pricing* assumes agents' degrees of risk aversion and prices are obtained by equating demand and supply in the financial markets.

1.1 Arbitrage pricing: the binomial model

Assume there are two states of nature that occur with probability p and $q = 1 - p$ and two assets each with price unity. Asset 1 ($A1$) pays 1 in state 1 and 1 in state 2. Similarly, asset 2 ($A2$) pays 0 in state 1 and 3 in state 2. We use the following notation: $A1$ pays $(1, 1)$ and $A2$ pays $(0, 3)$. Now assume that there is a third asset in this simple economy paying (2,3). Can we calculate its price? That is, what is the price $A3(0)$?

To obtain the value of $A3(t)$ at time $t = 0$ we set up a portfolio $\Pi(t = 0)$ consisting of a units of $A1$ and b units of $A2$ and deduce a and b such that $\Pi(t = 1) = A3(1)$. Hence, we must have that

$$\Pi^1(1) = 1\,a + 0\,b \tag{1}$$

and

$$\Pi^2(1) = 1\,a + 3\,b\,. \tag{2}$$

We require that $\Pi^1(1) = 2$ and $\Pi^2(1) = 3$, ie we replicate $A3$'s payoff. Therefore we must have that $a = 2$ and $b = 1/3$. Then, at time $t = 0$, $A3(0) = 2 + 1/3$.

- If the initial price of $A3$ was different, could we make a risk-less profit?
- If we had, instead of two, three possible states of the nature and $A1$ and $A2$ pay 1 in that state, what is the no-arbitrage value of $A3$ at time $t = 0$?

1.2 Pricing a Call option in a simple Binomial model

Now, we would like to apply the same arbitrage arguments to price of a European call. Recall that the payoff of a European call is given by

$$C^E(S, t = T; K, T) = \max(S_T - K, 0)\,.$$

Let us assume that there are two states of the world with probabilities p and $q = 1 - p$. Let the initial security price be S and suppose that if the 'up' state is revealed, with probability p, then the asset's value is $u\,S$ where u is a constant. Similarly, if the 'down' state is revealed the asset's value becomes $d\,S$ where d is a constant. Moreover, in this economy there is a bond with value B that pays a constant interest rate r per unit of time. In the up state the payoff of the call is

$$C_u^E = \max(u\,S - K, 0)$$

and in the down state the payoff is

$$C_d^E = \max(d\,S - K, 0)\,.$$

How can we find the arbitrage-free price of the call at time $t = 0$? Proceeding as before, we set a portfolio $\Pi(0)$ with an amount of cash B (i.e., we invest B in risk-less bonds) and short Δ amount of the underlying S. The value of the portfolio at inception is

$$\Pi(0) = B + \Delta\,S\,.$$

We must choose Δ so that the final payoff, in the two possible states of nature, matches the payoff of the European call we want to price by no-arbitrage. In other words,

$$\Delta\,u\,S + R\,B = C_u^E$$

and

$$\Delta\,d\,S + R\,B = C_d^E\,,$$

where the gross risk-free rate is denoted by $R = 1 + r$.

In matrix form, we solve the following system of equations

$$\begin{bmatrix} u\,S & R \\ d\,S & R \end{bmatrix} \begin{bmatrix} \Delta \\ B \end{bmatrix} = \begin{bmatrix} C_u^E \\ C_d^E \end{bmatrix},$$

so

$$\begin{bmatrix} \Delta \\ B \end{bmatrix} = \begin{bmatrix} u\,S & R \\ d\,S & R \end{bmatrix}^{-1} \begin{bmatrix} C_u^E \\ C_d^E \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \Delta \\ B \end{bmatrix} = \frac{1}{R(u\,S - d\,S)} \begin{bmatrix} R & -R \\ -d\,S & u\,S \end{bmatrix} \begin{bmatrix} C_u^E \\ C_d^E \end{bmatrix},$$

in other words

$$\Delta = \frac{C_u^E - C_d^E}{u\,S - d\,S}\,, \tag{3}$$

and the amount of cash we need when we set up the initial portfolio is

$$B = \frac{-d\,C_u^E + u\,C_d^E}{R\,(u-d)}\,.$$

Before proceeding we note an interesting point that will be of extreme importance throughout the course. The amount we are required to hold of the underlying, given by equation (23), can be seen as the change in the value of the option induced by a change in the underlying security S.

Hence, the portfolio at time $t = 0$ must have, by no-arbitrage, the same value that of the call option, i.e., $\Pi(0) = C^E(S, t = 0; K, 1)$.

$$\begin{aligned} C^E(S, t=0; K, 1) &= \Delta\,S + B \\ &= \frac{C_u^E - C_d^E}{u\,S - d\,S}\,S + \frac{-d\,C_u^E + u\,C_d^E}{R\,(u-d)} \\ &= \frac{1}{R}\left[\frac{R-d}{u-d}\,C_u^E + \frac{u-R}{u-d}\,C_d^E\right]. \end{aligned} \tag{4}$$

We must have $R > d$ and $u > R$ so the price of the option is financially plausible. In fact, if these restrictions did not hold we would have that either the risky asset yields a better return than the bond in all states of nature or vice-versa.

1.3 Risk-neutral valuation

The 'statistical' probabilities p and q for the underlying S to move up or down do not appear in (4). This is a very interesting fact because we are saying that if the probability of going up is $p = 0.999$ the price of the European call is the same as if the probability of the stock landing in the up state was $p = 0.01$. Initially one would expect that the call premium increases as the probability of expiration in-the-money increases. However, we showed with a very simple, yet powerful example, that statistical probabilities are irrelevant (well, this is true so long as $0 < p < 1$) when pricing options in the binomial model.

The value of the call in (4) can be seen as the discounted weighted average of the payoff at expiry. Moreover, given the financial restrictions of the parameters u, d, R we see that the weights

$$p^\star = \frac{R-d}{u-d} \tag{5}$$

and

$$q^\star = \frac{u-R}{u-d} \tag{6}$$

can be understood as risk-neutral probabilities. In other words, if we assume that agents are risk-neutral, pricing of instruments in this economy is performed as an expectation of the payoff using (5) and (6) as the (risk-neutral) probabilities.

Can we, in this risk-neutral world, calculate the discounted expected value of the stock price $\mathbb{E}[e^{-r}S(T)]$?

One way of doing this is by pricing a European call with $K = 0$. We had already established that the price of a call was given by

$$\begin{aligned} C^E(S, t=0; K, 1) &= \frac{1}{R}\left[\frac{R-d}{u-d}\,C_u^E + \frac{u-R}{u-d}C_d^E\right] \\ &= \frac{1}{R}\left[p^\star\,C_u^E + q^\star\,C_d^E\right] . \end{aligned}$$

But because $K = 0$, we must have that

$$\begin{aligned} C^E(S, t=0; K=0, 1) &= \frac{1}{R}\left[p^\star C_u^E + q^\star C_d^E\right] \\ &= \frac{1}{R}\left[p^\star u\, S + q^\star d\, S\right] \\ &= \frac{1}{R}\left[p^\star u\, S + (1-p^\star)\, d\, S\right] \\ &= \frac{1}{R}\left[p^\star u\, S + (1-p^\star)\, d\, S\right] \\ &= \frac{1}{R}\,\mathbb{E}^\star[S_T]\,, \end{aligned}$$

where S_T is the price of the asset at time T. Note that $C^E(S, t=0; K=0, 1) = S$, hence $S = \frac{1}{R}\,\mathbb{E}^\star[S_T]$, where the probabilities in the expectation are the risk-neutral probabilities derived above.

2 Early exercise

So far we have devoted this lecture to the pricing of European instruments. At this point we can ask ourselves whether we can price American options using the binomial framework

Take as an example an American put on a security that pays no dividend. The option payoff is given by

$$P_u^A = \max(K - u\, S, 0)\,,$$

$$P_d^A = \max(K - d\, S, 0)\,.$$

Without an early exercise opportunity in period 1, the value of the call would be

$$P^A(S, 0; K, 1) = \frac{1}{R}\left(p^\star P_u^A + q^\star P_d^A\right)\,.$$

However, it might be the case that $P^A(S, 0; K, 1) < K - S$. To see this, note that if $u\, S < K$, i.e.,, that the security is so deep in the money that it will always be exercised, then

$$P^A(S, 0; K, 1) = \frac{1}{R}\left(p^\star P_u^A + q^\star P_d^A\right) = \frac{K}{R} - S\,.$$

In multi-period models, say a binomial with n steps, the value of the American option at every node is the greater of its European equivalent or its intrinsic value (i.e., early exercise value).

3 Options

A **European (call/put) option** gives the right to the holder of the option to purchase/sell the underlying (for example a stock S) at a pre-specified time, called the expiration date T, for a pre-specified amount known as the strike price K.

An **American option** is like a European option with the difference that it can be exercised (ie buy or sell) at any time up until expiration T.

A European call offers its holder a right that they may or may not exercise. If at time T the underlying stock price S_T is greater that the strike price K, then a rational person will exercise the option and make $S_T - K$. In this case, the net profit is $S_T - K$ minus the premium paid for the option. On the other hand, if at time T the underlying security is below or equal to the strike price K, the holder of the option will not exercise it; therefore, the payoff is 0. In this case, the owner lost the initial premium paid for the option. The usual, and more compact, notation for this type of payoff is $\max(S_T - K, 0)$. Similarly, the payoff for a put option is $\max(K - S_T, 0)$.

One of the fundamental questions we will try to answer in this course is what is the 'fair' value of an option. The term vanilla options often refers to options with very simple payoffs like calls and puts. However, as markets have become more liquid over the years a great deal of different types of options have been developed. Options that have more complicated payoff structures than vanilla options are normally referred to as exotic options.

3.1 Pricing a Call option in a simple economy

Assume that there are two possible states. A stock S is trading at 100 and tomorrow it will either go up to 101 or down to 99. What is the value of a European call option with strike price $K = 100$?

What happens if the probability of landing in the bad state is $q = .95$?

4 Model independent properties of options

The valuation of options will depend on the stochastic properties we assume for the underlying. However, there are a number of properties that prices of vanilla options must satisfy regardless of the dynamics of the underlying security. For example, it is not difficult to convince ourselves that an American option is always at least as valuable as a European call written on the same underlying; with same expiry date and same strike price.

Let $C^E(S,t;K,T)$ and $C^A(S,t;K,T)$ respectively denote the prices of European and American calls and $P^E(S,t;K,T)$ and $P^A(S,t;K,T)$ denote the European and American puts.

Proposition 1 *Call prices satisfy the following inequalities*

$$C^A(S,t;K,T) \geq C^E(S,t;K,T); \tag{7}$$

$$C^A(S,t;K_1,T) \leq C^A(S,t;K_2,T), \qquad \textit{if } K_1 \geq K_2; \tag{8}$$

$$C^A(S,t;K,T_1) \geq C^A(S,t;K,T_2), \qquad \textit{if } T_1 \geq T_2; \tag{9}$$

$$C^A(S,t;K,T) \leq S; \tag{10}$$

$$C^A(0,t;K,T) = C^E(0,t;K,T) = 0. \tag{11}$$

Proof

The first three statements are obvious. For inequality (10) note that $C^A(S,t;K,T) \to S$ as $T \to \infty$. Then the following inequalities can be established: $C^A(S,t;K,T) \leq C^A(S,t;0,T) \leq C^A(S,t;0,\infty) = S$.

4.1 Early Exercise

An important result on American options is that a call written on an underlying that pays no dividends is never exercised early.

Proposition 2 *Let S be an underlying security that pays no dividends. Then an American call written on S is never exercised before expiry.*

Proof

First, we establish the following inequality

$$C^A(S,t;K,T) \geq S - K\,e^{-r\,(T-t)},$$

where r is the constant risk-free rate. By considering a portfolio with $C^A(S,t;K,T) - S + K\,e^{-r\,(T-t)}$, (i.e., long a call, short a share and with a bank deposit that at time T pays K) we observe the following. If the American call is exercised early the value of the portfolio is

$$S - K - S + K\,e^{-r\,(T-t)} = K\,(e^{-r\,(T-t)} - 1) < 0\,.$$

If we wait until time T we exercise if $S \geq K$ and the value of the portfolio is 0; if $S < K$ we do not exercise the option and the value of the portfolio is $K - S > 0$. Therefore we are better off waiting until T; hence, we have shown

$$C^A(S,t;K,T) \geq S - K\,e^{-r\,(T-t)}\,.$$

To show that an American call written on a stock that pays no dividend is never exercised early we observe that a call yields $S - K$ if exercised but $S - K \leq S - K\,e^{-r\,(T-t)}$ and $C^A(S,t;K,T) \geq S - K\,e^{-r\,(T-t)} \geq S - K$.

The analogous result for the American put is not true. Assume that the stock price falls so far that

$$(K - S_t)\, e^{r\,(T-t)} > K - S_T\,, \tag{12}$$

but we can sharpen this result by assuming that at expiry $S_T = 0$ hence if at time $t < T$ the stock price S_t is low enough so that

$$(K - S_t)\, e^{r\,(T-t)} > K\,, \tag{13}$$

it is worthwhile exercising early.

4.2 Put-Call parity

There is a classical result relating the value of European options.

Proposition 3 *Put-Call parity for European options*

$$C^E(S,t;K,T) - P^E(S,t;K,T) = S - K\, e^{-r\,(T-t)}\,. \tag{14}$$

5 Black–Scholes

5.1 Modelling returns

The first attempt to model stock prices with independent Gaussian increments was over a century ago (Bachelier). One of the shortcomings of Bachelier's model was to model stock levels, instead of returns. Modeling prices with Brownian motion has the implicit assumption that price can become negative, which cannot be the case because the shareholders are protected (i.e., limited liability). A company goes bankrupt and the shares become worthless but not of negative value.

It was not until much later when Samuelson proposed to model stock returns, instead of levels, in the following way

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t\,, \tag{15}$$

where μ is known as the drift and $\sigma \geq 0$ as the volatility.

Exercise 1 *Let the stock returns follow (15). What process is followed by S_t?*

To answer the question we will make use of Ito's lemma. But first let's take a naive approach to the question and integrate both sides of (15) to obtain

$$\begin{aligned} \int_0^t \frac{dS_u}{S_u} &= \int_0^t \mu \, du + \sigma \int_0^t dW_u \\ \ln S_t / S_0 &= \mu \, t + \sigma \, W_t \, . \end{aligned}$$

Hence (it seems that)

$$S_t = S_0 \, e^{\mu \, t + \sigma \, W_t} \, . \tag{16}$$

Now let us check if our result is correct. Instead of integrating as if we were in a deterministic setting we use Ito's lemma. Let $f = \ln S$, what is df? First rewrite (15) as

$$dS_t = S_t \, \mu \, dt + S_t \, \sigma \, dW_t \, . \tag{17}$$

Next, use Ito's lemma with $\mu(S,t) = \mu \, S_t$ and $\sigma(S,t) = \sigma \, S_t$, i.e., use (**??**) with

$$f = \ln S, \qquad \frac{\partial f}{\partial S} = 1/S, \qquad \frac{\partial^2 f}{\partial S^2} = -1/S^2, \qquad \frac{\partial f}{\partial t} = 0$$

to obtain

$$\begin{aligned} d(\ln S_t) &= \left(S_t \, \mu \, \frac{1}{S} - \frac{1}{2} \, S_t^2 \, \sigma^2 \, \frac{1}{S_t^2} \right) dt + S_t \, \sigma \, \frac{1}{S_t} \, dW_t & (18) \\ &= \left(\mu - \frac{1}{2} \, \sigma^2 \right) dt + \sigma \, dW_t \, . & (19) \end{aligned}$$

Hence, by integrating both sides, we obtain

$$S_t = S_0 e^{(\mu - \frac{1}{2} t \sigma^2) \, t + \sigma \, W_t} \, . \tag{20}$$

Note that the naive solution was 'nearly' correct but it was lacking the $\frac{1}{2}\sigma^2 t$ term.

5.2 The Black–Scholes partial differential equation

Now we have all the necessary ingredients to derive the Black–Scholes partial differential equation (PDE). We will assume that the stock follows a geometric Brownian motion, or equivalently, that the returns follow an arithmetic Brownian motion

$$dS_t = S_t\,\mu\,dt + S_t\,\sigma\,dW_t\,, \tag{21}$$

where dW is the increment of a standard Brownian motion, $\mu \geq 0$ and $\sigma \geq 0$.

Assume that we want to price a European-style option, $V(S,t;K,T)$, written on the stock S with terminal payoff $V(S,T;K,T) = V(S,T)$.

To obtain its arbitrage free value we proceed as in the previous lectures and set a portfolio $\Pi(S,t)$ long the option and short Δ amount of S

$$\Pi(S,t) = V(S,t;K,T) - \Delta\,S_t.$$

The idea is to obtain Δ so that the portfolio is hedged at every point in time. Recall that when we priced options within a binomial framework we calculated the amount of shares that would guarantee that the portfolio would replicate the payoff of the option in every state of nature. The main difference here is that there are an infinite number of states of nature because the stochastic dynamics is given by $W(t)$. Hence, for a small time step we would like to hold (or be short, depending on the sign of Δ) Δ number of shares S so that within that time interval the portfolio is (perfectly) hedged. Moreover, because the dynamics of the stock price are continuous we will be interested in infinitesimally small time increments dt, therefore we must find the change in Π as time evolves. This is given by

$$d\Pi_t = dV_t - \Delta\,dS_t\,.$$

The change in S we already know because it is given by equation (21). The dV term must be calculated using Ito's Lemma because the value of the option depends on the uncertain stock price. Therefore, using Ito's lemma we obtain

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \mu\,S_t\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2}\right)\,dt + \sigma\,S_t\,\frac{\partial V}{\partial S}\,dW_t - \Delta\,(S\,\mu\,dt + S\,\sigma\,dW_t)\,.$$

Rearranging terms we have

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \mu\, S\, \frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2} - \Delta\,\mu\, S \right)\, dt + \left(\frac{\partial V}{\partial S} - \Delta \right)\, S\sigma\, dW_t\,. \quad (22)$$

It is interesting to note that 'all' the randomness in the evolution of the portfolio is captured in the last term $\left(\frac{\partial V}{\partial S} - \Delta\right)\, S\,\sigma\, dW$. We can choose, at every instant in time,

$$\Delta = \frac{\partial V}{\partial S}\,, \quad (23)$$

so the change in the the value of the portfolio is deterministic. Therefore substituting (23) into (22) yields

$$\begin{aligned} d\Pi_t &= \left(\frac{\partial V}{\partial t} + \mu\, S\, \frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2} - \frac{\partial V}{\partial S}\,\mu\, S \right)\, dt \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2} \right)\, dt\,. \end{aligned} \quad (24)$$

A very important step in the derivation of the Black–Scholes PDE stems from the fact that we choose Δ so that there is no randomness in the evolution of Π; hence we must have that the portfolio grows like a riskless bond, i.e.,

$$d\Pi_t = r\,\Pi_t\, dt\,.$$

Putting these results together we have

$$\begin{aligned} r\,\Pi\, dt &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2} \right)\, dt \\ r\left(V - \frac{\partial V}{\partial S} S \right)\, dt &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2} \right)\, dt \\ r\, V &= \frac{\partial V}{\partial t} + r\, S\, \frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2}\,, \end{aligned} \quad (25)$$

which is the Black–Scholes PDE.

5.3 The Black–Scholes PDE

In the previous lecture we showed that by forming a suitable portfolio, long a European option and short an amount of stock $P(S,t) = V(S,t) - \Delta S$, we derived an equation the Black–Scholes pricing equation

$$r\,V = V_t + r\,S\,V_S + \frac{1}{2}\,\sigma^2\,S^2\,V_{SS}\,, \tag{26}$$

where subscripts denote partial differentiation, ie $V_t = \partial V(S,t)/\partial t$ and so on.

Before proceeding we might want to ask ourselves a few questions.

- How general is equation (53)?
- Can we price any type of European option? How?
- Can we price any American option?
- How come the drift μ of the returns process is nowhere to be seen?
- If μ seems to have been replaced by the risk-free rate r should not we see a different σ?
- What happens if the stock pays a dividend?
- What happens if I can only rebalance my portfolio at discrete points in time?
- Is it really possible to assume continuous hedging?
- etc.

When we derived equation (53) we made no specific assumptions about the type of European option we were pricing. Hence, we should be able to price any European-style option written on a stock that follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t \tag{27}$$

with final payoff $V(S,T)$. It is very important to note that what makes a great difference is the payoff (i.e., boundary condition) when solving (53). For example if we solve (53) subject to $V(S,t) = \max(K - S, 0)$ we obtain the price of a European put. Similarly, if we let $V(S,t) = \max(S - K, 0)$ we obtain the price of a European call. We can also think of more general payoffs such as power calls where $V(S,t) = (\max(S - K, 0))^n$ for $n = 1, 2, 3, \ldots$ and power puts (defined in a similar way).

However, at this point, perhaps the most important question to ask is: how do we solve (53)?

Equation (53) can be reduced, after straightforward change of variables, to the classical Heat Equation, see the textbook by Wilmott, Howison and Dewynne. The Heat Equation, also known as the Diffusion Equation, describes the flow of heat in a continuous medium; in our case what 'diffuses' is the value of the option.

Let us start with a call option that satisfies

$$C_t^E + \frac{1}{2}\,\sigma^2\,S^2\,C_{SS}^E + r\,S\,C_S^E - rC^E = 0\,, \tag{28}$$

subject to $V(S_T, T) = \max(S_T - K, 0)$ and

$$C(0,t) = 0, \qquad C(S,t) \sim S \quad \text{as } S \to \infty.$$

The last two conditions are given by the financial nature of the problem.

Our next task is to reduce equation (28), by a suitable change of variables, to the heat equation and then solve it. The intuition is more or less this. First, we note that in the heat equation problem we have an initial condition, i.e., initial amount of heat, that diffuses as time goes by. In our case we have a final condition, the payoff, and what we are really interested in is obtaining the 'initial' condition that would fit our problem. However, the approach we take is to change time so our final condition becomes the initial condition, hence we let

$$t = T - \tau/\frac{1}{2}\,\sigma^2\,.$$

It will also be convenient to get rid of the S coefficients and we can do it by letting

$$S = Ke^x,$$

or in other words we let $x = \ln S/K$. Note that we divide S by the strike price and we do this because we also get rid of the units in which is measured, ie pounds; it does not make sense to take the natural logarithm of dollars. Finally, we also let $C = Kv(x,\tau)$ and, after some algebra,

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\,\frac{\partial v}{\partial x} - k\,v\,, \tag{29}$$

where $k = r/\frac{1}{2}\sigma^2$. We also need to change variables in the boundary conditions to get

$$v(x,0) = \max(e^x - 1, 0).$$

Note that our condition at time $t = T$ has become an initial condition in our 'new' problem.

We must take another step for our equation (29) to look like the heat equation. We let

$$v(x,\tau) = e^{\alpha x + \beta \tau} u(x,\tau)\,,$$

for some constants α and β to be found, then differentiation gives

$$\beta\, u + \frac{\partial u}{\partial \tau} = \alpha^2\, u + 2\,\alpha\, \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - k\, u\,.$$

Next, choose

$$\beta = \alpha^2 + (k-1)\,\alpha - k,$$

to eliminate the u term. Moreover, the choice

$$2\,\alpha + (k-1) = 0$$

eliminates the $\partial u/\partial x$ term. Therefore by choosing

$$\alpha = -\frac{1}{2}\,(k-1), \qquad \beta = -\frac{1}{4}\,(k+1)^2$$

we obtain

$$v(x,\tau) = e^{\frac{1}{2}\,(K-1)\,x - \frac{1}{4}\,(k+1)^2\,\tau} u(x,\tau)\,,$$

where

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for} -\infty < x < \infty,\ \tau > 0,$$

with

$$u(x,0) = \max(e^{\frac{1}{2}\,(k+1)x} - e^{\frac{1}{2}\,(k-1)\,x}, 0)\,. \tag{30}$$

The solution to the problem is given by the solution to the heat equation, hence

$$u(x,\tau) = \int_{-\infty}^{\infty} u(x,0)\, e^{-\frac{(x-s)^2}{4\tau}}\, ds\,. \tag{31}$$

In other words, we are weighting the initial condition by the function $e^{-\frac{(x)^2}{4\tau}}$. Note that this function is basically the probability density function of a normal distribution $N(0, 2\tau)$.

Now we can evaluate (31), and after some algebra, we obtain

$$u(x,\tau) = e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}\,\Phi(d_1) - e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau}\,\Phi(d_1)\,,$$

where $\Phi(y)$ is the normal cumulative density function and where

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$\begin{aligned} d_1 &= d_2 + \sigma\sqrt{T-t} \\ &= \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,. \end{aligned} \tag{32}$$

The last step we must take is to go back to the original variables, i.e.,

$$x = \ln(S/K), \qquad \tau = \frac{1}{2}\sigma^2(T-t), \qquad C = K\,v(x,\tau),$$

to get

$$C(S,t;K,T) = S_t\,\Phi(d_1) - e^{-r(T-t)}\,K\,\Phi(d_2)\,. \tag{33}$$

5.4 Valuation with Intermediate Income Flows

So far we have considered de pricing of vanilla options when the underlying security pays no dividends. In this section we relax this assumption and include continuous and discrete dividends.

By continuous dividends we mean that the asset pays a flow per unit of time equal to $D_0 S_t$ for D_0 constant. By discrete dividends we mean (known) lump sums amount of money that are payed to the holder of the stock at (known) prescribed times in the future.

5.5 Constant proportional dividends

Suppose that in a time dt the underlying asset pays out a dividend $D_0 S dt$ where D_0 is a constant. This payment is independent of time except through the dependence on S. The dividend yield is defined as the proportion of the asset price paid out per unit of time this way. Thus the dividend $D_0 S dt$ represents a constant and continuous dividend yield D_0.

By arbitrage we have that the asset price must 'leak' value at every time step hence the random walk must reflect this in the following way

$$\frac{dS_t}{S_t} = (\mu - D_0)\, dt + \sigma\, dW_t\,. \tag{34}$$

Now, to price a European option on a security that follows (34) we proceed as in the non-dividend case and form a portfolio $P(S,t) = V - \Delta S$. The Black–Scholes pricing equation becomes

$$r\, V \;=\; \frac{\partial V}{\partial t} + (r - D_0)\, S\, \frac{\partial V}{\partial S} + \frac{1}{2}\, \sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2} \tag{35}$$

subject to the relevant boundary conditions. Note that one must be careful in the first step when calculating the change in the value of the portfolio $dP_t = dV_t - \Delta\, dS_t - D_0\, \Delta\, S_t\, dt$.

If we were to price a European call we can proceed as above transform our PDE (35) to the heat equation and solve as above. Then the price of the call would be given by

$$C^E = e^{-D_0(T-t)} S_t \Phi(d_{10}) - e^{-r\,(T-t)} K \Phi(d_{20}),$$

where

$$d_{10} \;=\; \frac{\ln(S_t/K) + (r - D_0 + \frac{1}{2}\sigma^2)(T-t)}{\sigma\, \sqrt{T-t}}$$

and

$$d_{20} \;=\; d_{10} - \sigma\, \sqrt{T-t}.$$

5.6 Futures and Forwards

We can apply the proportional income flow model to pricing options of futures and forwards. A forward contract is an agreement to buy a security at a given price (fixed now) at some specified future date. There is no initial payment in a forward in that one agrees to exchange the asset and make payments only at a future time.

If S_t and F_t are the price of the security (without dividend) and the forward respectively for a contract maturing in $T - t$. There are several ways of deriving the forward price. Consider first the party who is short the contract, and so must deliver the asset at time T. Although he does not know what the value of S will be at T this does not matter. He can borrow S_t when the contract begins, purchase the stock and at expiry he can use the amount F to pay off the loan. Assuming that the risk-free rate r is constant, the loan will cost $S_t\, e^{r\,(T-t)}$. The forward price must therefore be given by

$$F = S_t\, e^{r\,(T-t)}\,. \tag{36}$$

5.7 General risk-neutral valuation

More generally, we could derive the following expression to price **any** European-style option with payoff $V(S,T)$

$$V(S,t) = \frac{e^{-r\,(T-t)}}{\sqrt{2\,\sigma^2\,\pi\,(T-t)}} \int_0^\infty V(u,T)\, e^{-\left(\ln(u/S)-(r-\frac{1}{2}\,\sigma^2)\,(T-t)\right)^2/(2\,\sigma^2\,(T-t))}\, \frac{du}{u}\,. \tag{37}$$

Therefore all we need is the parameters of the SDE that drives the underlying, the payoff of the derivative and we calculate its value by evaluating (37).

A Various results we use

Toolbox

The expected value of a lognormal random variable will appear very often throughout the course. It is easy to show that for a random variable $X \sim N(0,1)$ the expected value

$$\mathbb{E}[e^{\theta X}] = e^{\frac{1}{2}\theta^2}. \tag{38}$$

Assume the risk-neutral dynamics of the stoch price are given by

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t^*\,.$$

Let us try to solve the expectation

$$\begin{aligned} C^E &= \mathbb{E}_t^\star[e^{-r\,(T-t)} \max(S_T - K, 0)] \\ &= e^{-r\,(T-t)} \int_{-\infty}^{\infty} \max(S_T - K, 0)\, f(\phi)\, d\phi\,, \end{aligned} \tag{39}$$

where the density function

$$f(\phi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} \tag{40}$$

is the probability density function of the standard random variable $\phi \sim N(0,1)$. Moreover, we denote the cumulative density function by

$$\Phi(d) = \int_{-\infty}^{d} f(\phi) d\phi.$$

Use Ito's lemma to write

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\phi\sqrt{T-t}}\,;$$

here, we wrote the Brownian motion $\int_t^T dW_u$ as the random variable $\phi\,\sqrt{T-t}$ with $\phi \sim N(0,1)$.

Next,

$$\begin{aligned}
C^E &= e^{-r\,(T-t)} \int_{-\infty}^{\infty} \max(S_T - K, 0) f(\phi) d\phi, \\
&= e^{-r\,(T-t)} \int_{-d_2}^{\infty} (S_T - K) f(\phi) d\phi \\
&= e^{-r\,(T-t)} S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{\sigma\phi\sqrt{T-t}} e^{-\frac{1}{2}\phi^2} d\phi \\
&\quad - e^{-r\,(T-t)} K \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}\phi^2} d\phi, \\
&= S_t \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}\sigma^2(T-t)} e^{\sigma\phi\sqrt{T-t}} e^{-\frac{1}{2}\phi^2} d\phi \\
&\quad - e^{-r\,(T-t)} K \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{1}{2}\phi^2} d\phi, \\
&= S_t \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(\phi-\sigma\sqrt{T-t})^2} d\phi - e^{-r\,(T-t)} K \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{1}{2}\phi^2} d\phi, \\
&= S_t \frac{1}{\sqrt{2\pi}} \int_{-d_2-\sigma\,\sqrt{T-t}}^{\infty} e^{-\frac{1}{2}y^2} dy - e^{-r\,(T-t)} K \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{1}{2}\phi^2} d\phi, \\
&= S_t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2+\sigma\,\sqrt{T-t}} e^{-\frac{1}{2}y^2} dy - e^{-r\,(T-t)} K \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{1}{2}\phi^2} d\phi, \\
&= S_t \Phi(d_1) - e^{-r\,(T-t)} K \Phi(d_2),
\end{aligned} \tag{41}$$

where

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\,\sqrt{T-t}},$$

$$\begin{aligned}
d_1 &= d_2 + \sigma\,\sqrt{T-t} \\
&= \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\,\sqrt{T-t}}.
\end{aligned} \tag{42}$$

A The Heat Equation

Solving the Black–Scholes PDE reduces to solving the classical heat equation problem. Below we give a concise (based on Strauss) intuitive proof of the solution to the heat equation. See also the textbook (Wilmott, Howison, Dewynne) for a discussion.

We want to solve the following PDE

$$u_t = \kappa\, u_{xx}, \qquad x \in \mathbb{R}, \quad 0 < t < \infty \tag{43}$$

$$u(x,0) = \phi(x)\,. \tag{44}$$

We will solve it for a particular $\phi(x)$ and then build the general solution from this particular one.

We will show (see Strauss) that the solution to (43) is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy\,, \qquad t > 0$$

where

$$S(x,t) = \frac{1}{2\sqrt{\pi\,\kappa t}}\, e^{-\frac{x^2}{4\,\kappa\,t}}\,.$$

Before proceeding note that the function $S(x,t)$ is the probability density function of a normal random variable $N \sim (0, 2\kappa t)$.

Invariance Principles of the Heat Equation

1. The translate $u(x-y,t)$ of any solution $u(x,t)$ is another solution, for any fixed y.

   $$u_t(x-y,t) = u_t(x-y,t)$$

   say $z = x - y$ then

   $$u_x = u_z\,\frac{dz}{dx}\,,\ \text{hence } u_x = u_z \text{ and } u_{xx} = u_{zz}\,.$$

2. Any derivative (u_x or u_{xx}, etc) of a solution is again a solution. Let $V = u_x$, then $V_x = u_{xx}$, $V_{xx} = u_{xxx}$ and $V_t = u_{xt}$. We must then have

   $$V_x = \frac{\partial}{\partial x} u_{xx} = \frac{\partial}{\partial x} u_t = V_t\,.$$

3. A linear combination of (43) is again a solution

   $$(a\,u^1 + b\,u^2)_{xx} = (a\,u^1 + b\,u^2)_t\,.$$

4. An integral of solutions is again a solution. Thus if $S(x,t)$ is a solution of (43) then (by 1.) $S(x-y,t)$ is also a solution and so is

$$V(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,g(y)\,dy$$

for any function $g(y)$ as long as the integral converges.

5. If $u(x,t)$ is a solution of (43) so is the scaled function $u(\sqrt{a}\,x, a\,t)$ for any $a > 0$.

Now we proceed to finding a solution to (43). Look for a particular solution

$$Q(x,t),\ \text{with } Q(x,0) = 1 \text{ for } x > 0 \text{ and } Q(x,0) = 0 \text{ for } x < 0\,. \tag{45}$$

STEP 1

Guess $Q(x,t) = g(p)$ where $p = \frac{x}{\sqrt{4\kappa t}}$. We guess this form because property 5 says that equation (43) does not see the scaling $x \to \sqrt{a}\,x$, $t \to a\,t$. Clearly (45) possesses this property.

$$p = \frac{x}{\sqrt{4\,\kappa\,t}} \overset{x\to\sqrt{a}\,x, t\to at}{\longrightarrow} \frac{x}{\sqrt{4\,\kappa\,t}}\,.$$

STEP 2

$$Q_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2\,t}\,\frac{x}{\sqrt{4\,\kappa\,t}}\,g'(p)\,, \tag{46}$$

$$Q_x = \frac{1}{\sqrt{4\,\kappa\,t}}g'(p), \qquad \text{and} \qquad Q_{xx} = \frac{1}{4\,\kappa\,t}g''(p). \tag{47}$$

Hence we obtain the ODE

$$g'' + 2\,p\,g\,g' = 0\,. \tag{48}$$

Solve using the integrating factor $\int 2\,p\,dp = e^{p^2}$, $g'(p) = C_1\,e^{-p^2}$ and

$$Q(x,t) = g(p) = C_1 \int e^{-p^2}\,dp + C_2\,.$$

STEP 3

$$Q(x,t) = C_1 \int_0^{\frac{x}{\sqrt{4\kappa t}}} e^{-p^2} dp + C_2, \qquad \text{for } t > 0.$$

If $x > 0$

$$\lim_{t\to 0} Q = C_1 \int_0^{\infty} e^{-p^2}\, dp + C_2 = \frac{C_1\sqrt{\pi}}{2} + C_2\,,$$

if $x < 0$

$$\lim_{t\to 0} Q = C_1 \int_0^{-\infty} e^{-p^2}\, dp + C_2 = \frac{-C_1\sqrt{\pi}}{2} + C_2$$

with

$$C_1 = \frac{1}{\sqrt{\pi}}, \qquad C_2 = \frac{1}{2}\,.$$

Hence

$$Q(x,t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\frac{x}{\sqrt{4\kappa t}}} e^{-p^2}\, dp\,, \qquad t > 0.$$

STEP 4

Define $\partial Q/\partial x$, given ϕ define

$$u(x,t) = \int_{\infty}^{-\infty} S(x-y,t)\phi(y)dy, \qquad t > 0.$$

Now we can claim that $u(x,t)$ is a solution to (43). To verify that $u(x,0) = \phi(x)$

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)dy \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)dy && (49) \\ &= \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy - Q(x-y,t)\phi(y)|_{-\infty}^{\infty}. && (50) \end{aligned}$$

Assume that the limits vanish. In particular let us assume that $\phi(y) = 0$ for $|y|$ large. Hence

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)dy.$$

Now $Q(x-y,0) = 1$ when $x - y > 0$. Hence

$$\begin{aligned} u(x,0) &= \int_{-\infty}^{x} \phi'(y)dy && (51) \\ &= \phi(x). && (52) \end{aligned}$$

Hence

$$S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi \kappa t}} e^{-\frac{x^2}{4\kappa t}}.$$

If we want to solve the Black–Scholes PDE

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \tag{53}$$

subject to the relevant boundary conditions, we must change variables to transform equation (53) into the heat equation (43) that we know how to solve.