C2.6 Introduction to Schemes Sheet 2

Hilary 2024

(1) (A) Let $(X, \mathcal{O}_X)$ be a scheme $U \subseteq X$ an open subset, $s \in \mathcal{O}_X(U)$ a section. Show:

(a) $\{x \in U : s_x = 0 \in \mathcal{O}_{X,x}\} \subseteq U$ is open;

(b) $\{x \in U : s(x) = 0 \in \kappa(x)\} \subseteq U$ is closed.

(2) (B) Fill in the details of the proof of the following theorem (see lecture notes).

Theorem. *For all schemes X and rings R, there is a natural bijection*

$$\mathrm{Maps}_{\mathsf{Sch}}(X, \operatorname{Spec} R) \cong \mathrm{Maps}_{\mathsf{Ring}}(R, \mathcal{O}_X(X)).$$

(3) (B) Let X be a topological space. Suppose that we are given an open cover $\{U_\alpha\}_\alpha$ together with sheaves $\mathcal{F}_\alpha$ on each U_α and isomorphisms

$$\varphi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} \mathcal{F}_\beta|_{U_\beta \cap U_\alpha},$$

satisfying the *cocycle condition* $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

Show that the $\{\mathcal{F}_\alpha\}_\alpha$ glue to a sheaf $\mathcal{F}$ on X.

(4) (B) Use exercise 3 to show that, given schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ together with open subsets $U \subseteq X$, $V \subseteq Y$, and an isomorphism

$$(U, \mathcal{O}_X|_U) \xrightarrow{\sim} (V, \mathcal{O}_Y|_V),$$

one can perform gluing to obtain a scheme whose underlying topological space is $(X \sqcup Y)/(U \sim V)$, and whose structure sheaf is the glued structure sheaf.

(5) (B) Prove that the following schemes are not affine:

(a) $\mathbb{A}^2_{\mathbb{Z}} \setminus \{(0,0)\}$, viewed as an open subscheme of $\mathbb{A}^2_{\mathbb{Z}}$;

(b) The projective line: glue $\mathbb{A}^1_{\mathbb{Z}}$ and $\mathbb{A}^1_{\mathbb{Z}}$ by identifiying the open subsets $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ and $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\} = \operatorname{Spec} \mathbb{Z}[u, u^{-1}]$ via the isomorphism induced by $t \leftrightarrow u^{-1}$.

(c) The line with two origins: glue $\mathbb{A}^1_{\mathbb{Z}}$ and $\mathbb{A}^1_{\mathbb{Z}}$ by identifiying the open subsets $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ and $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\} = \operatorname{Spec} \mathbb{Z}[u, u^{-1}]$ via the isomorphism induced by $t \leftrightarrow u$.

(6) (B) Let k be a field with $\operatorname{char} k \neq 2$. Show that $\operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$ is an integral scheme. Show that its preimage in $\operatorname{Spec} k[\![x,y]\!]/(y^2 - x^2 - x^3)$ is reducible. Briefly discuss the geometric intuition.

(7) (B) Construct the scheme $\mathbb{P}^n_{\mathbb{Z}}$ by glueing $n+1$ copies of $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[y_0, \ldots, y_n]$, where for the i^{th} copy, we use coordinates $y_1 = x_0/x_1, \ldots, y_n = x_n/x_i$, (omitting x_i/x_i).