Axiomatic Set Theory: Problem sheet 2

A.

1. (ZF*) Define a "natural" ordinal exponentiation using the recursion theorem for ordinals, and show that for all ordinals α, β and γ, $\alpha^{(\beta+\gamma)} = \alpha^\beta\alpha^\gamma$, and $\alpha^{(\beta\cdot\gamma)} = (\alpha^\beta)^\gamma$. Show also that $2^\omega = \omega$.

2. (ZF*) Prove that $(V, \in)$ satisfies the Axiom of Unions and the Axiom of Infinity.

3. (ZF*) Let $\alpha \in \mathbf{On}$ and suppose that $a \in V_\alpha$ and $b \subseteq a$. Prove that $b \in V_\alpha$.

B.

4. (ZF*) Suppose $F : \mathbf{On} \to \mathbf{On}$ is a class term satisfying:
(1) $\alpha < \beta \to F(\alpha) < F(\beta)$ (for $\alpha, \beta \in On$)
(2) $F(\delta) = \bigcup_{\alpha<\delta} F(\alpha)$ (for limit ordinals δ).
Prove that for all $\alpha \in \mathbf{On}$ there exists $\beta \in \mathbf{On}$ such that $\beta > \alpha$ and $F(\beta) = \beta$ (ie. F has arbitrarily large fixed points). What is the smallest non-zero fixed point of the term $F : On \to \mathbf{On}$ defined by $F(x) = \omega.x$ (for $x \in \mathbf{On}$)?

5. (ZF*) Prove that the axiom of foundation is equivalent to $\forall x(x \in V)$.

6. (ZF*) Later in the course we shall be concerned with those formulas whose truth does not depend on which transitive class they are interpreted in. More precisely, let A be a transitive class. A formula $\phi(v_1, \ldots, v_n)$ (*without* parameters) of LST is called A-*absolute* if for any $a_1, \ldots, a_n \in A$, $\phi(a_1, \ldots, a_n)$ holds (ie. $(V^*, \in) \models \phi(a_1, \ldots, a_n)$) iff $\phi(a_1, \ldots, a_n)$ holds in A (ie. $(A, \in) \models \phi(a_1, \ldots, a_n)$). Prove that the following statements (or the natural formulas of LST which these translate) are A-absolute, for any transitive class A:
(i) $v_1 \subseteq v_2$ (ii) $v_1 = \bigcup v_2$ (iii) $v_1 = \{v_2, v_3\}$ (iv) $v_1 = v_2 \cup \{v_2\}$.

C.

7. Show that "$v_1 = \wp v_2$" is not ω-absolute. (Note that ω is a transitive class.)