Further Partial Differential Equations (2024)
Problem Sheet 2

Questions 2 and 3 will be marked.

1. **Similarity solutions in higher dimensions**

Consider the equation for the thickness of liquid on a vertical substrate derived in question 1 with the addition of surface tension smoothing in the transverse direction:

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\rho g}{3\mu}\frac{\partial}{\partial \hat{z}}\left(\hat{h}^3\right) + \frac{\gamma}{3\mu}\frac{\partial}{\partial \hat{x}}\left(\hat{h}^3\frac{\partial^3 \hat{h}}{\partial \hat{x}^3}\right) = 0, \tag{1}$$

where γ is the coefficient of surface tension.

(a) Non-dimensionalize the system using

$$\hat{h} = Hh, \qquad \hat{z} = Lz, \qquad \hat{x} = Lx, \qquad \hat{t} = Tt,$$

to obtain the dimensionless version of (1),

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial z}\left(h^3\right) + \frac{1}{3}\frac{\partial}{\partial x}\left(h^3\frac{\partial^3 h}{\partial x^3}\right) = 0, \tag{2}$$

for suitably chosen H, L and T, which you should find.

(b) Assume first that the thickness is independent of transverse direction, x. Seek a similarity solution of the form $h = f(\eta)$ where $\eta = z/t^\alpha$ and find the equation that is satisfied by f and the required value of the parameter α.

(c) By solving the differential equation found in (b), show that $f = (z/t)^{1/2}$.

(d) Now assume that the thickness depends on x, z and t. Seek a solution of the form $h = f(\eta)g(\nu)$ where f is the function given in (c) and $\nu = xt^\beta z^\delta$. By substituting this ansatz into (2) show that g satisfies the equation

$$-12g + 12g^3 + 3\nu g' - 9\nu g^2 g' + 24g^2 g' g''' + 8g^3 g'''' = 0$$

where primes denote differentiation, for suitably chosen β and δ that you should find.

Solution

(a) Choosing

$$L = \left(\frac{\gamma H}{\rho g}\right)^{1/3}, \qquad T = \frac{\mu \gamma^{1/3}}{(\rho g)^{4/3} H^{5/3}},$$

yields the required dimensionless equation. Note that H remains arbitrary.

(b) This is identical to in the lecture notes and very similar to question 2 but this provides a stepping stone to the more complicated solution. Choosing $\alpha = 1$ leads to the differential equation

$$\left(f^2 - \eta\right) f' = 0. \tag{3}$$

(c) Again, this part is identical to in the lecture notes and the solution follows straightforwardly by dividing (3) by f' and integrating.

(d) This is the trickier part and requires some careful algebra. Substituting the ansatz $h = f(\eta)g(\nu)$ where $f = \sqrt{\eta}$, $\eta = z/t$ and $\nu = xt^\beta z^\delta$ gives

$$g - 2\beta\nu g' - 2g^2 g' \left(\delta\nu + t^{4\beta-1/2} z^{4\delta+3/2} g'''\right) - \frac{1}{3} g^3 \left(3 + 2t^{4\beta-1/2} z^{4\delta+3/2} g''''\right) = 0. \tag{4}$$

From here, we immediately see that we must choose $\beta = 1/8$ and $\delta = -3/8$. In doing so, the differential equation (4) becomes

$$-12g + 12g^3 + 3\nu g' - 9\nu g^2 g' + 24 g^2 g' g''' + 8 g^3 g'''' = 0,$$

as required.

2. **Possible similarity solutions**
Consider the partial differential equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + x^{\alpha} f^{\beta}\right), \tag{5}$$

subject to the boundary conditions

$$f \to 0 \qquad \text{as} \quad x \to \pm\infty. \tag{6}$$

Suppose that $\beta > \alpha \geq 0$ and that f is suitably well behaved so that $xf \to 0$ as $x \to \pm\infty$.

(a) Show that

$$\int_{-\infty}^{\infty} f(x,t)\,\mathrm{d}x \tag{7}$$

is a constant.

(b) Using (5)–(7), show that a similarity solution of the form $f = t^a g(\eta)$ with $\eta = x/t^b$ exists for this system provided

$$\alpha = \beta - 2 \tag{8}$$

and given values of a and b.

(c) Show that g satisfies the ordinary differential equation

$$\frac{1}{2}\eta g + g' + \eta^{\beta-2} g^{\beta} = 0. \tag{9}$$

(d) In the case when $\beta = 2$, show that we can write (9) in the form

$$\left(\frac{\eta}{2g} + \frac{g'}{g^2}\right) \mathrm{e}^{-\eta^2/4} = -\mathrm{e}^{-\eta^2/4} \tag{10}$$

and so by recognizing an exact differential on the left-hand side, show that the solution is

$$g = \frac{\mathrm{e}^{-\eta^2/4}}{\sqrt{\pi}\,(\coth(G/2) + \mathrm{erf}(\eta/2))}, \tag{11}$$

where

$$G = \int_{-\infty}^{\infty} g(\eta)\,\mathrm{d}\eta \tag{12}$$

is a constant.

Solution

(a) Integration of (5) and use of (6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} f(x,t)\,\mathrm{d}x = 0$$

and so

$$\int_{-\infty}^{\infty} f(x,t)\,\mathrm{d}x$$

is a constant.

(b) Try $f(x,t) = t^a g(\eta)$ with $\eta = x/t^b$. Equation (7) indicates that we must choose $a = -b$. Substituting the ansatz $f(x,t) = t^a g(\eta)$ with $\eta = x/t^b$ into (5) we obtain the ordinary differential equation

$$-bg - b\eta g' = t^{1-2b} g'' + \alpha x^{\alpha-1} t^{b+1-\beta b}\left(g^{\beta} + \beta\eta g^{\beta-1} g'\right) \tag{13}$$

and so we must choose

$$b = \frac{1}{2}, \qquad\qquad \alpha = \beta - 2 \tag{14}$$

for the equation to be in similarity variables.

(c) These choices can be imposed to equation (13) and then we may integrate this equation once. The constant of integration is determined to be zero by applying the conditions as $\eta \to \pm\infty$ and the regularity conditions given in the question assumptions that imply that ηg and $\eta^{\beta-2} g^{\beta}$ both tend to 0 as $\eta \to \pm\infty$. This leads to the required (9).

(d) Simple rearrangement and multiplication of both sides by $\mathrm{e}^{-\eta^2/4}$ gives (10). This can be written as

$$\frac{\partial}{\partial\eta}\left(\frac{\mathrm{e}^{-\eta^2/4}}{g}\right) = \mathrm{e}^{-\eta^2/4}. \tag{15}$$

Integration gives

$$g = \frac{\mathrm{e}^{-\eta^2/4}}{A + \sqrt{\pi}\,\mathrm{erf}(\eta/2)}; \tag{16}$$

A is determined by the integral constraint:

$$\int_{-\infty}^{\infty} g\,\mathrm{d}\eta = G, \text{ say,} \tag{17}$$

which may be evaluated and rearranged to give

$$A = \sqrt{\pi}\coth\left(G/2\right). \tag{18}$$

The result then follows.

3. **Outwardly radial spreading in a porous medium**

Consider the radial spreading of a fixed volume of liquid in a porous medium. The height $\hat{h}$ of the liquid is governed by the equation

$$\phi \frac{\partial \hat{h}}{\partial \hat{t}} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \hat{h} \hat{Q} \right) = 0, \qquad \hat{Q} = -\frac{\rho g k}{\mu} \frac{\partial \hat{h}}{\partial \hat{r}} \tag{19}$$

where $\hat{r}$ and $\hat{t}$ denote respectively the radial coordinate and time and $\hat{Q}$ is the flux; ρ denotes the density of the fluid, g acceleration due to gravity, k the permeability, ϕ the porosity and μ the fluid viscosity.

(a) Write down the equation that expresses conservation of mass.

(b) By choosing suitable non-dimensionalization show that the system may be reduced to one that contains no physical parameters.

(c) By finding the appropriate form of the similarity solution, show that the problem can be reduced to solving the following ordinary differential equation system,

$$(\eta f f')' + \frac{1}{4}\eta^2 f' + \frac{1}{2}\eta f = 0, \tag{20}$$

$$\int_0^{\eta_f} \eta f(\eta)\, \mathrm{d}\eta = 1, \tag{21}$$

$$f'(0) = 0, \tag{22}$$

$$f(\eta_f) = 0, \tag{23}$$

where you should define the functions $\eta = \eta(r,t)$, $\eta_f = \eta_f(r,t)$ and $f = f(\eta, t)$.

(d) By rescaling $s = \eta/\eta_f$ and $g = f/\eta_f^2$ find the ordinary differential equation that is satisfied by g and a condition for η_f in terms of g.

(e) By performing a local analysis show that the conditions at the front are

$$g(1) = 0, \qquad g'(1) = -\frac{1}{4}. \tag{24}$$

(f) Hence show that the solution is given by $g(s) = (1 - s^2)/8$, $\eta_f \approx 2^{5/4}$.

(g) Based on the results of this analysis, is this a similarity solution of the first or second kind? What physical feature of the problem indicates that it is a similarity solution of this kind?

(a) Conservation of mass is expressed via the equation

$$2\pi \int_0^{\hat{r}_f} \hat{r}\hat{h}(\hat{r},\hat{t})\,\mathrm{d}\hat{r} = \hat{V}, \tag{25}$$

where $\hat{V}$ is the volume of liquid. We non-dimensionalize via the following scalings

$$\hat{r} = \hat{r}_0 r, \quad \hat{r}_f = \hat{r}_0 r_f, \quad \hat{t} = \left(\frac{2\pi r_0^4 \mu\phi}{\rho g k \hat{V}}\right) t, \quad \hat{h} = \left(\frac{\hat{V}}{2\pi \hat{r}_0^2}\right) h, \quad \hat{Q} = \left(\frac{\rho g k \hat{V}}{2\pi\mu \hat{r}_0^3}\right) Q. \tag{26}$$

Here, the radial scale $\hat{r}_0$ is arbitrary. The governing equations and mass conservation become

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rhQ) = 0, \tag{27}$$

$$Q = -\frac{\partial h}{\partial r}, \tag{28}$$

$$\int_0^{r_f} rh\,\mathrm{d}r = 1. \tag{29}$$

(b) Try $h = t^\alpha f(\eta)$ with $\eta = r/t^\beta$. Substituting into the governing equations and volume constraint show that we require $\alpha = -1/2$ and $\beta = 1/4$. This leads to the required system (20) and (21) while the conditions (22) and (23) represent respectively symmetry at the centre at the behaviour at the front.

(c) Applying the scalings given leads to the new system

$$(sgg')' + \frac{1}{4}s^2 g' + \frac{1}{2}sg = 0, \tag{30}$$

$$\eta_f = \left(\int_0^1 sg(s)\,\mathrm{d}s\right)^{-1/4}. \tag{31}$$

(d) We perform a local analysis by scaling $s = 1 + \delta\xi$, $g = \delta G$ to obtain

$$(G(\xi)G'(\xi))' + \frac{1}{4}G'(\xi) = 0 \tag{32}$$

$$\Rightarrow G'(\xi) = -\frac{1}{4}, \tag{33}$$

which in terms of the original variables gives the required result $g'(1) = -1/4$.

(e) It is straightforward to substitute the given form for g and show that this is a solution. Alternatively, one could try $g(s) = a + bs^2$ and find the required values for a and b. The value of η_f is obtained by substituting the solution $g(s) = (1 - s^2)/8$ into (31).

(f) This is a similarity solution of the first kind as the form of the similarity solution was determined by substituting into the governing equation. The problem has no natural lengthscale, which is an indicator that a similarity solution exists.