# String Theory 1

Lecture #8

Final remarks : We have gone over the OCQ of the Polyakov action which describes a free field theory in 1+1 dims

In this scheme we discussed how to construct the Hilbert space of physical states. The consistency of the quantum spectrum (no ghosts) requires

$$a = 1 \quad \& \quad D = 26$$

(together with some input from the interacting string to rule out $0 < a \leq 1$ & $1 \leq D \leq 25$)

For the critical string at low level we found the following

NN

- ground state (both open & closed strings): tachyon
- open string at level 1:

NN

$|\xi;k\rangle = \xi\cdot\alpha_{-1}|0;k\rangle$ is a massless state } photon

$\xi\cdot k = 0$ $\qquad \xi_\mu \sim \xi_\mu + \partial_\mu \lambda$

- closed string level $N=\tilde{N}=1$
  - ▸ symmetric traceless tensor $\boxed{\gamma_{\mu\nu}}$ metric perturbations (dynamical gravity)

    $\gamma_{\mu\nu}(x) \sim \gamma_{\mu\nu}(x) + \partial_\mu S_\nu + \partial_\nu S_\mu$
  - ▸ antisymmetric tensor $\boxed{b_{\mu\nu}}$ Ramond-Kalb 2-form gauge field

    $b_{\mu\nu}(x) \sim b_{\mu\nu}(x) + \partial_\mu S_\nu - \partial_\nu S_\mu \;, \quad S_\mu \to S_\mu + \partial_\mu \lambda$
  - ▸ a scalar $\boxed{\varphi}$ dilaton

ST II: superstrings retain these features and ∄ tachyon.

# Chapter 3 Interactions

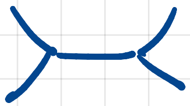
## 3.1 Generalities

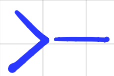
QFT:

- to understand interactions one adds non-linear terms to the action

  ↳ doesn't work for the string because anything you try to add breaks gauge invariance.

- scattering amplitudes → Feynman diagrams

  eg  etc

  interactions encoded at vertices say  3 point int.

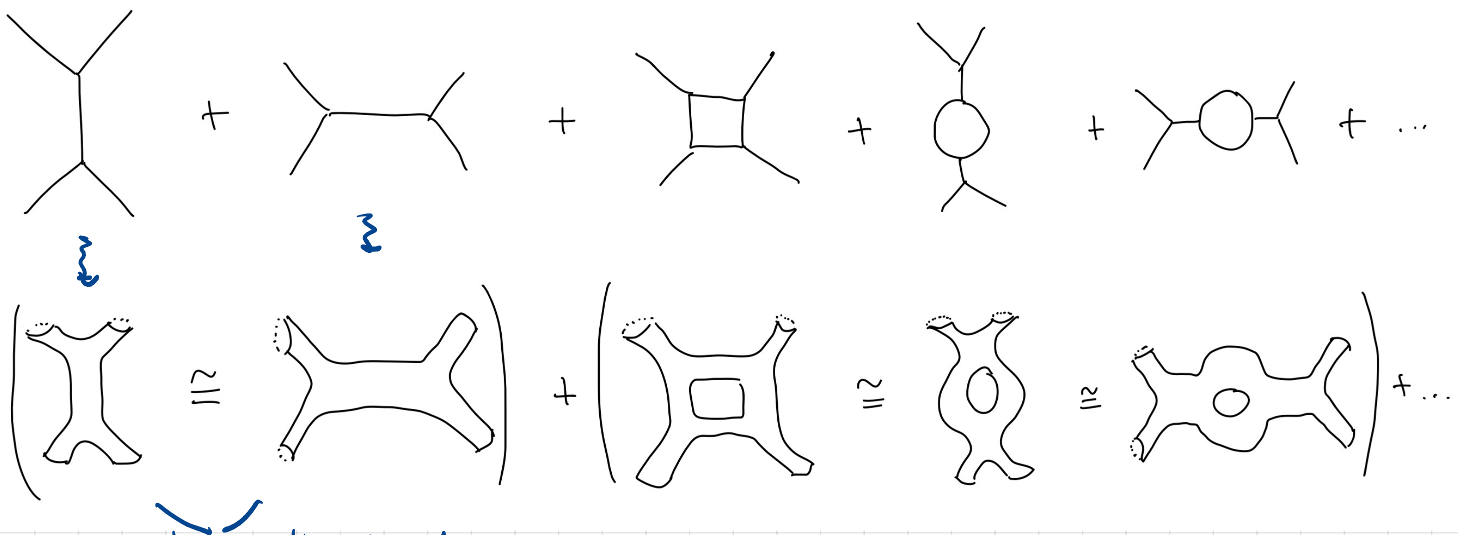
  ↳ in string theory this is replaced by, for instance

   or  : no such vertices!

summing all possible Feynman diagrams with fixed external legs (and integrating over the positions of all vertices), one recovers an asymptotic series expression for the scattering amplitude.

QFT Feynman diagrams

stringy version



topologically equiv.

sums of diagrams of strings branching and joining

- each locally look like a 2dim manifold
- different particle-diagrams become topologically equivalent string diagrams.

diagrams for both open & closed strings

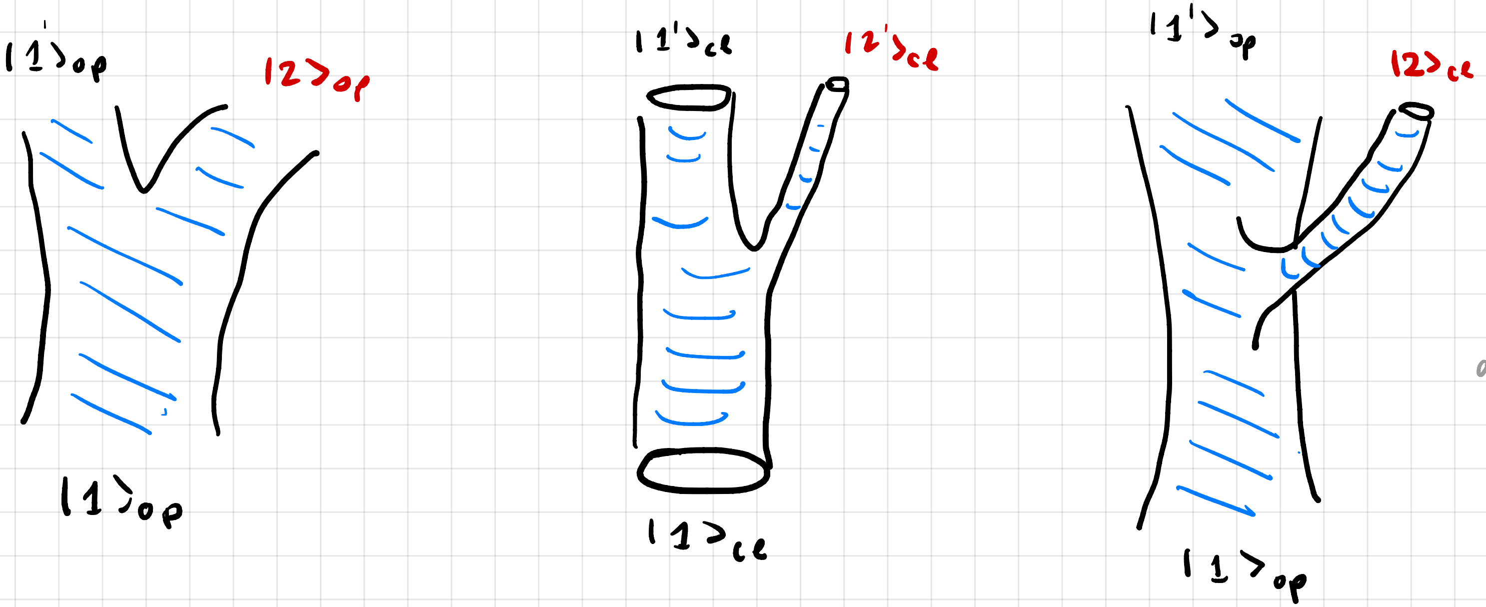
- solid black lines < OS: boundaries of the WS; CS: only the ends of the WS are boundaries

# In string theory :

We want to compute for example the amplitude of a given configuration of quantised strings at an initial time to evolve into a new configuration at a later time

$A$: conf of q-strings $(t_i)$ ⟶ conf of q-strings $(t_f)$

This means we need to describe the branching and joining of quantised strings




open string $|1\rangle_{op}$ which splits into two $|1'\rangle_{op}$ & $|2\rangle_{op}$

open string branches into an OS & a CS

**Problem** : it is not known how to do this ....

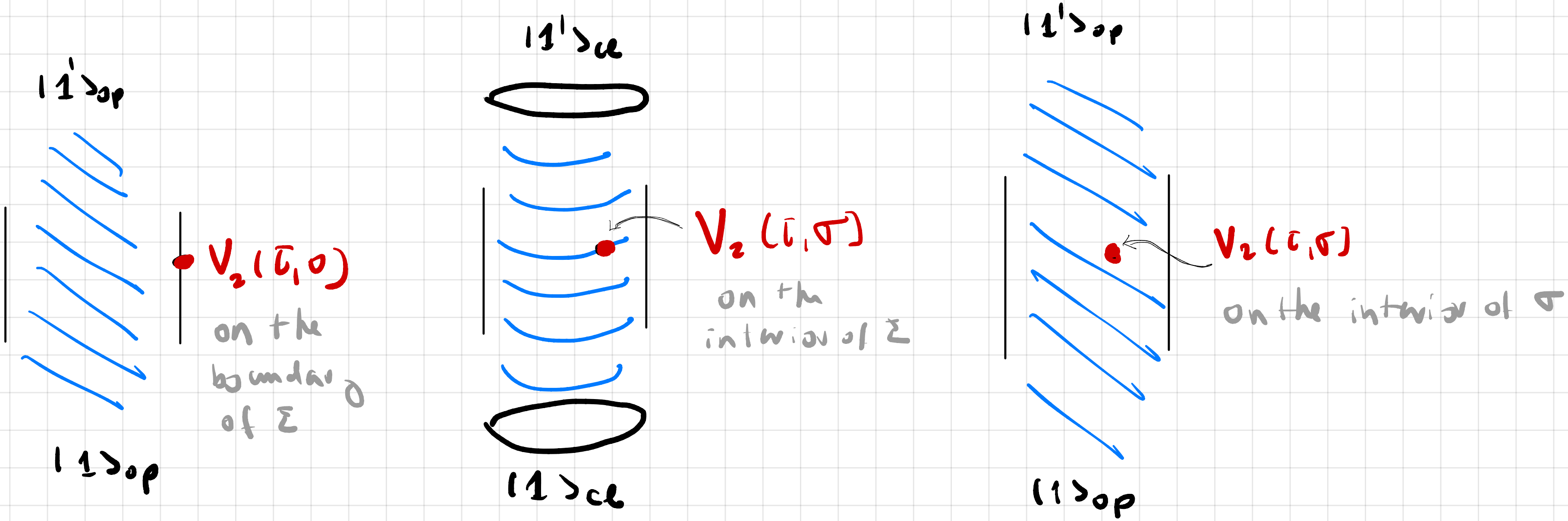
as we do not have a second quantised theory

all we have at hand is the phys / the null physical states of a string

So, we need to work with the first quantised formalism

Suppose $|2\rangle$ is a physical state which is emitted/absorbed.

We describe the emission\absorption of a quantum state (say $|2\rangle$) from a fixed string WS by the action of a local operator or Vertex operator.



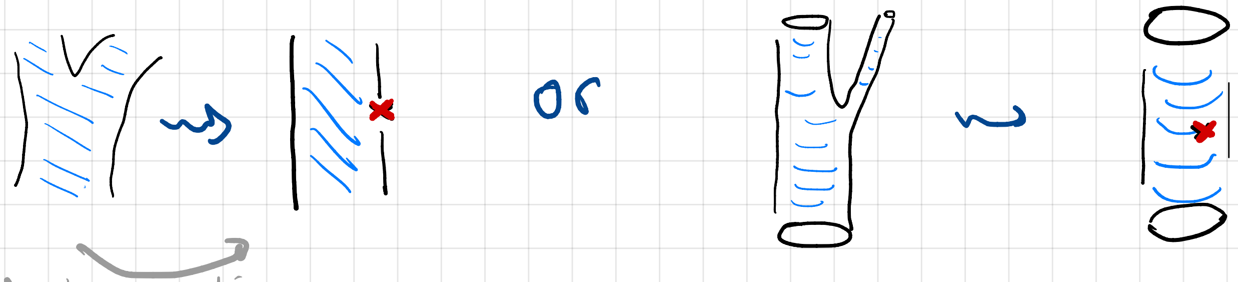
"Hand waving" explanation:

$|2\rangle$ is a physical state emitted/absorbed which is "quantum sized" (mass squared and width of order $\hbar$), so it behaves (almost) like a "point particle" in a classical limit.

Not very satisfactory!

<u>Instead</u>: we will describe this (*) process in this section

- Wick rotation of the world sheet (Lorentzian signature → Euclidean signature) more later
- conformal transformation



(*)

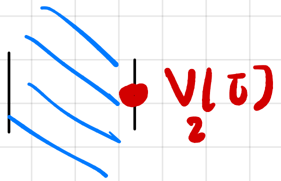
Wick rotation
+ conformal transformation
so can choose this a gauge choice!

## 3.2 Vertex operators : Introduction

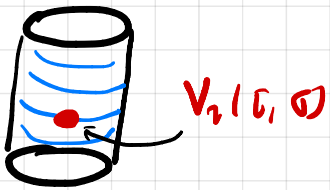
**Require:** $\boxed{\text{two}}$ key requirements

1. time evolution on the worldsheet is a gauge transformation ⇒ position of the vertex operator should not be meaningful.

open string vertex operator $\xrightarrow{\text{what matters is}}$ $\int d\tau \, \underbrace{V_2(\tau)}_{\substack{\downarrow \\ \text{inserted on the boundary}}}$




closed string vertex operator $\rightarrow$ $\int d\tau \, d\sigma \, \underbrace{V_2(\tau, \sigma)}_{\substack{\downarrow \\ \text{inserted on the interior}}}$



② absorption & emission string operators to map

$$\mathcal{H}_{\text{phys}} \longrightarrow \mathcal{H}_{\text{phys}}$$

and

$$\mathcal{H}_{\text{null}} \longrightarrow \mathcal{H}_{\text{null}}$$

Let's see how this works for the open strings first

Consider the action of the vertex operator on a physical state $|\phi_{\text{phys}}\rangle$, that is

(local) boundary operator at (say) $\sigma = 0$

$$\int d\tau \; V(\tau, 0) \; |\phi_{\text{phys}}\rangle$$

$\mathcal{H}_{phys} \xrightarrow{V} \mathcal{H}_{phys}$ Applying the physical state conditions:

$$m \geq 0 : \quad (L_m - a\,\delta_{m,1}) \left[ \int d\bar{\tau}\; V(\bar{\tau},0) \,|\phi_{phys}\rangle \right] = \int d\bar{\tau}\; [L_m, V(\bar{\tau},0)]\,|\phi_{phys}\rangle$$

$$= 0 \quad \text{if} \quad \boxed{[L_m, V(\bar{\tau},0)] = \partial_{\bar{\tau}} \big( \text{local operator} \big) \qquad m \geq 0}$$

$\int d\bar{\tau}\, \partial_{\bar{\tau}}(\quad) = (\quad)_f - (\quad)_i$

local operator at past & future infinity

for any reasonable physics

$\mathcal{H}_{null} \xrightarrow{V} \mathcal{H}_{null}$

$$m \geq 1 : \quad \underbrace{\int d\bar{\tau}\; V(\bar{\tau},0) \overbrace{[L_{-m}|\phi\rangle]}^{\text{null}}} = \int d\bar{\tau} \Big\{ \underbrace{[V(\bar{\tau},0), L_{-m}]}_{\substack{\text{need this to vanish up to a} \\ \text{total derivative}}} |\phi\rangle + \underbrace{L_{-m} V(\bar{\tau},0)|\phi\rangle}_{\substack{\text{null} \\ \langle \psi_{phys} | L_{-m} V(\bar{\tau},0) | \phi\rangle = 0}} \Big\}$$

This is null if

$$\boxed{[V(\bar{\tau},0), L_{-m}] = \partial_{\bar{\tau}}(\text{local op}), \qquad m \geq 1}$$

// Remark (added to answer a student's question)

We have seen that if $[L_m, V(\bar{z}, 0)] = \partial_{\bar{z}}(\text{local op}) \ \forall m$

then $\mathcal{H}_{\text{phys}} \overset{V}{\longleftrightarrow} \mathcal{H}_{\text{phys}}$, $\mathcal{H}_{\text{null}} \overset{V}{\longleftrightarrow} \mathcal{H}_{\text{null}}$

The converse is also true: suppose $V(\bar{z})$ is an operator st $\mathcal{H}_{\text{phys}} \overset{V}{\longleftrightarrow} \mathcal{H}_{\text{phys}}$, $\mathcal{H}_{\text{null}} \overset{V}{\longleftrightarrow} \mathcal{H}_{\text{null}}$.

Let $|\phi\rangle$ be a physical (or null) state. Then as $V(\bar{z})|\phi\rangle$ is physical (or null), we have

$$0 = \int d\bar{z}\,(L_m - \delta_{m,0})\,V(\bar{z})|\phi\rangle = \int d\bar{z}\,[L_m - \delta_{m,0}, V(\bar{z})]|\phi\rangle$$

$$= \int d\bar{z}\,[L_m, V(\bar{z})]|\phi\rangle \qquad \forall m$$

Therefore $[L_m, V(\bar{z})] = \partial_{\bar{z}}(\text{local operator})$ //

What happens to VOs under conformal transformation?

Moreover, conformal transformations of the open string are of the form $\tau \to \tilde{\tau}(\tau)$. We want (morally)

$\int V(\tau, 0)\, d\tau$ to be invariant under $\tau \to \tilde{\tau}$ (as emitted/absorbed state must be inv under $\tau \to \tilde{\tau}$)

$$\int V(\tau, 0)\, d\tau \longrightarrow \int \underbrace{\tilde{V}(\tilde{\tau}, 0)}\, d\tilde{\tau} = \int V(\tau, 0)\, d\tau$$

$$= \int \underbrace{V(\tau, 0) \frac{d\tau}{d\tilde{\tau}}}\, d\tilde{\tau}$$

so we want $V(\tau, 0) \longrightarrow \tilde{V}(\tilde{\tau}, 0) = V(\tau, 0) \dfrac{d\tau}{d\tilde{\tau}}$ under C transfs

**Definition**: an operator $A(\bar{\tau})$ is a primary operator of weight $h$ if under the transformation $\bar{\tau} \to \tilde{\bar{\tau}}(\bar{\tau})$ it transforms as

$$\boxed{A(\bar{\tau}) \longrightarrow \tilde{A}(\tilde{\bar{\tau}}) = A(\bar{\tau}) \left( \frac{d\bar{\tau}}{d\tilde{\bar{\tau}}} \right)^{h}}$$

For an operator with $h=1$

$$\int \tilde{A}(\tilde{\bar{\tau}})\, d\tilde{\bar{\tau}} = \int A(\bar{\tau}) \frac{d\bar{\tau}}{d\tilde{\bar{\tau}}} d\tilde{\bar{\tau}} = \int A(\bar{\tau})\, d\bar{\tau}$$

ie the integrated operator is conformally invariant.

(study the infinitesimal case)

For infinitesimal transformations $\tau \longrightarrow \tilde{\tau} = \tau + \epsilon(\tau)$

we have

$$A(\tau) \to \tilde{A}(\tilde{\tau}) = A(\tau) \Big( \underbrace{1 + h \frac{d\tau}{d\tilde{\tau}}}_{} \Big) \qquad \left(\frac{d\tau}{d\tilde{\tau}}\right)^h$$

OTOH (LHS)

$$\tilde{A}(\tilde{\tau}) = \tilde{A}(\tau + \epsilon) \underset{\text{taylor}}{=} \tilde{A}(\tau) + \epsilon \partial_\tau A(\tau) + O(\epsilon^2)$$

Then we find the variation of $A$ at $\tau$

$$\delta A(\tau) = \tilde{A}(\tau) - A(\tau) = -\epsilon \partial_\tau A - h(\partial_\tau \epsilon) A$$

$$= -\partial_\tau(\epsilon A) - (h-1)\partial_\tau \epsilon A$$

This is a total derivative when $h = 1$

So the conformal transformation (from now on set $\ell = \begin{cases} 2\pi & CS \\ \pi & OS \end{cases}$)

$$\tau \to \tilde{\tau} = \tau + \epsilon(\tau) \quad \text{with} \quad \epsilon = -e^{im\tau}$$

corresponds to

$$\delta A(\tau) = e^{im\tau}\left(-i\partial_\tau A + h m A\right)$$

Recall
$\{L_m, \phi(\tau)\}_{PB} = \delta_{\epsilon_m}\phi(\tau)$
$\{\cdot,\cdot\}_{PB} \to i[\cdot,\cdot]$

So the action of the Virasoro operators on the ops $\phi$ is

$$-i\,\delta\phi(\tau) = \boxed{[L_m, \phi(\tau)] = e^{im\tau}(-i\partial_\tau + mh)\,\phi(\tau)}$$

Equivalently, this is the condition for $\phi$ to have conformal weight $h$.

For $h=1$: $[L_m, \phi(\tau)] = \partial_\tau(-i e^{im\tau}\phi(\tau))$ ✓ TD

The problem is to identify primaries of weight 1 which correspond to the physical states in the string Hilbert space.

Then we use this to compute string amplitudes!

End of lecture #8

↳ Next

- def of closed string vertex op
- state ↔ vertex correspondence