B5.6 Nonlinear Dynamics, Bifurcations and Chaos

Sheet 3 — HT 2024

Solutions to all problems in Sections A and C

Section A: Problems 1, 2 and 3

1. Consider the ODE system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1\,(1-x_1) + x_1^2 - x_1^3 - 2\,x_1\,x_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1\,x_2 - \mu\, x_2$$

where $\mu \in (0,1)$ is a parameter.

(a) Find and classify all bifurcations of the ODE system for $0 < \mu < 1$.

(b) Sketch the phase plane for $\mu = 1/2$ and $\mu = 1/4$.

Solution:

(a) The equilibrium points satisfy

$$0 = \mu\, x_1\,(1-x_1) + x_1^2 - x_1^3 - 2\,x_1\,x_2$$
$$0 = x_1\,x_2 - \mu\, x_2$$

Consequently, the second equation implies $x_1 = \mu$ or $x_2 = 0$. Substituting into the first equation, we obtain the critical points

$$\mathbf{x}_{c1} = \left[\mu, \mu - \mu^2\right], \qquad \mathbf{x}_{c2} = [0,0], \qquad \mathbf{x}_{c3} = [1,0] \qquad \text{and} \qquad \mathbf{x}_{c4} = [-\mu, 0].$$

The Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \mu + 2(1-\mu)x_1 - 3x_1^2 - 2x_2 & -2x_1 \\ x_2 & x_1 - \mu \end{pmatrix},$$

giving

$$D\mathbf{f}(\mathbf{x}_{c1}) = \begin{pmatrix} \mu - 3\mu^2 & -2\mu \\ \mu - \mu^2 & 0 \end{pmatrix}, \qquad D\mathbf{f}(\mathbf{x}_{c2}) = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix},$$

$$D\mathbf{f}(\mathbf{x}_{c3}) = \begin{pmatrix} -\mu - 1 & -2 \\ 0 & 1-\mu \end{pmatrix} \qquad \text{and} \qquad D\mathbf{f}(\mathbf{x}_{c4}) = \begin{pmatrix} -\mu - \mu^2 & 2\mu \\ 0 & -2\mu \end{pmatrix}.$$

Consequently, $\mathbf{x}_{c2}$ and $\mathbf{x}_{c3}$ are saddles and $\mathbf{x}_{c4}$ is a stable node for all considered values of parameter μ, *i.e.* for $0 < \mu < 1$. The eigenvalues corresponding to matrix $D\mathbf{f}(\mathbf{x}_{c1})$ satisfy

$$\lambda^2 + (3\mu^2 - \mu)\,\lambda + 2\mu\,(\mu - \mu^2) = 0$$

Radek Erban: erban@maths.ox.ac.uk

giving

$$\lambda_\pm = \frac{\mu - 3\mu^2 \pm \mu\sqrt{9\mu^2 + 2\mu - 7}}{2}.$$

If $\mu < 7/9$, then we have two complex conjugate eigenvalues

$$\lambda_\pm = \frac{\mu - 3\mu^2}{2} \pm i\,\frac{\mu\sqrt{|9\mu^2 + 2\mu - 7|}}{2}.$$

The real part is positive for $\mu < 1/3$ and negative for $\mu > 1/3$. We have a pair of purely imaginary eigenvalues at the bifurcation point $\mu = 1/3$, when

$$\lambda_\pm = \pm\frac{2}{3\sqrt{3}}\,i$$

and the stability of the critical point $\mathbf{x}_{c1}$ changes at the bifurcation point $\mu = 1/3$. We introduce new variables

$$y_1 = x_1 - \mu\,, \qquad y_2 = \sqrt{3}\left(x_2 - \mu + \mu^2\right), \qquad \overline{\mu} = \mu - \frac{1}{3}\,.$$

Then the ODE system transforms to

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}t} &= \left(-\overline{\mu} - 3\,\overline{\mu}^2\right) y_1 - \frac{6\,\overline{\mu} + 2}{3\sqrt{3}}\,y_2 - \frac{2}{\sqrt{3}}\,y_1\,y_2 - \left(4\,\overline{\mu} + \frac{1}{3}\right) y_1^2 - y_1^3 \\
\frac{\mathrm{d}y_2}{\mathrm{d}t} &= \frac{2 + 3\,\overline{\mu} - 9\,\overline{\mu}^2}{3\sqrt{3}}\,y_1 + y_1\,y_2
\end{aligned}$$

which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = M(\overline{\mu})\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -\frac{2}{\sqrt{3}}\,y_1\,y_2 - \left(4\,\overline{\mu} + \frac{1}{3}\right) y_1^2 - y_1^3 \\ y_1\,y_2 \end{pmatrix}, \tag{1}$$

where matrix $M(\overline{\mu})$ is

$$M(\overline{\mu}) = \begin{pmatrix} -\overline{\mu} - 3\,\overline{\mu}^2 & -\frac{6\,\overline{\mu}+2}{3\sqrt{3}} \\ \frac{2+3\,\overline{\mu}-9\,\overline{\mu}^2}{3\sqrt{3}} & 0 \end{pmatrix}$$

Close to the bifurcation point $\overline{\mu} = 0$, matrix $M(\overline{\mu})$ has eigenvalues

$$\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$$

where

$$\alpha(\overline{\mu}) = -\,\frac{\overline{\mu} + 3\,\overline{\mu}^2}{2}$$

and

$$\omega(\overline{\mu}) = \frac{2}{3\sqrt{3}}\sqrt{1 + \frac{27\,\overline{\mu}}{6} - \frac{27\,\overline{\mu}^2}{16} - \frac{378\,\overline{\mu}^3}{16} - \frac{243\,\overline{\mu}^4}{16}}$$

which implies

$$\alpha(0) = 0\,, \qquad \omega(0) = \frac{2}{3\sqrt{3}}\,, \qquad \alpha'(0) = -\frac{1}{2}\,, \qquad \text{and} \qquad \omega'(0) = \frac{\sqrt{3}}{2}.$$

Radek Erban: erban@maths.ox.ac.uk

Substituting $\overline{\mu} = 0$ into equation (1), the system reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix}$$

where

$$h_1(y_1, y_2) \;=\; -\frac{2}{\sqrt{3}}\, y_1\, y_2 \,-\, \frac{1}{3}\, y_1^2 \,-\, y_1^3$$

and

$$h_2(y_1, y_2) \;=\; y_1\, y_2\,.$$

Evaluating the partial derivatives at the origin $\mathbf{0} = [0, 0]$, we get

$$\begin{aligned} a(0) \;&=\; \frac{1}{16}\left(\frac{\partial^3 h_1}{\partial y_1^3} + \frac{\partial^3 h_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 h_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 h_2}{\partial y_2^3}\right) + \frac{1}{16\,\omega(0)}\left[\frac{\partial^2 h_1}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_1}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\right)\right. \\ &\qquad\left. - \frac{\partial^2 h_2}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_2}{\partial y_2^2}\right) - \frac{\partial^2 h_1}{\partial y_1^2}\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\frac{\partial^2 h_2}{\partial y_2^2}\right] \\ &=\; \frac{1}{16}(-\,6 \,+\, 0 \,+\, 0 \,+\, 0) + \frac{1}{16\,\omega(0)}\left[-\frac{2}{\sqrt{3}}\left(-\frac{2}{3} + 0\right) - 1\,(0+0) - 0 + 0\right] \\ &=\; -\frac{1}{4}\,. \end{aligned}$$

Since $a(0) < 0$, we have a supercritical Hopf bifurcation at $\overline{\mu} = 0$, *i.e.* the original system has a supercritical Hopf bifurcation at $\mu = 1/3$. The normal form is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{2}\,\overline{\mu}\,r \,-\, \frac{1}{4}\,r^3 \,+\, \ldots$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{2}{3\sqrt{3}} \,+\, \frac{\sqrt{3}}{2}\,\overline{\mu} \,+\, \ldots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} > 0$, *i.e* for $\mu > 1/3$ and unstable for $\overline{\mu} < 0$, *i.e* for $\mu < 1/3$. A stable limit cycle is born with amplitude

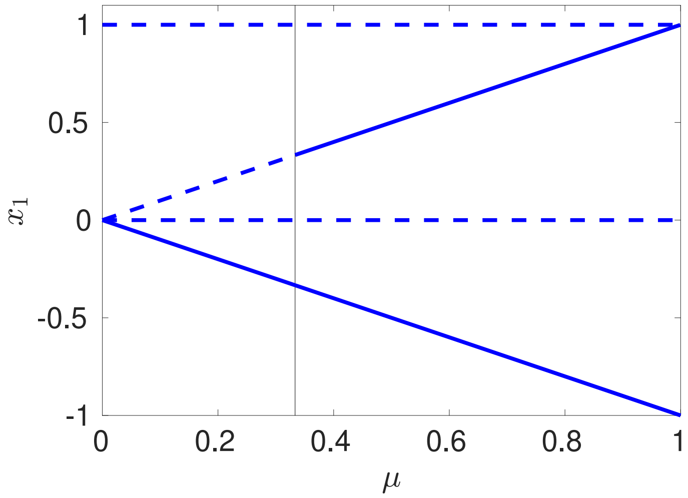
$$\sqrt{2\left(\frac{1}{3} - \mu\right)}$$

and period $3\,\pi\,\sqrt{3}$ at the bifurcation point for $\mu < 1/3$. The limit cycle can be approximated by

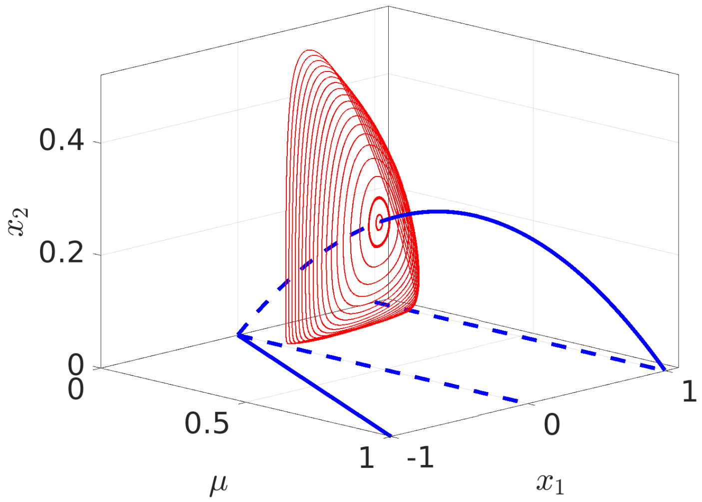
$$y_1^2 + y_2^2 = 2\left(\frac{1}{3} - \mu\right)$$

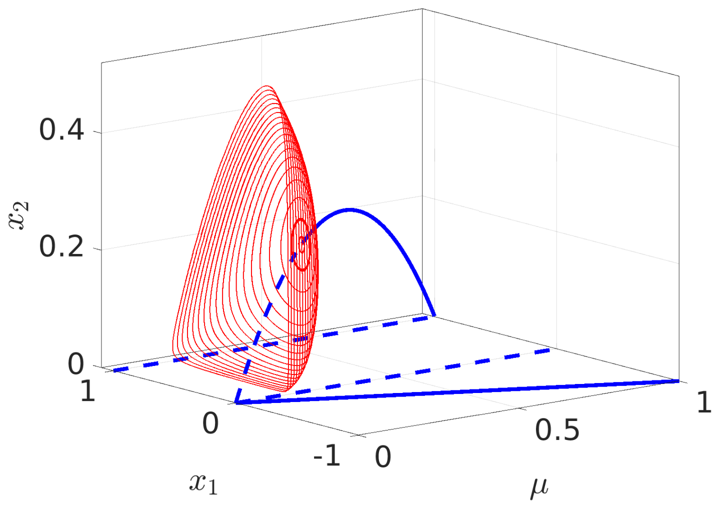
which corresponds to an ellipse in x_1 and x_2 variables.

The bifurcation diagram can be drawn in the μ–x_1 plane as follows:

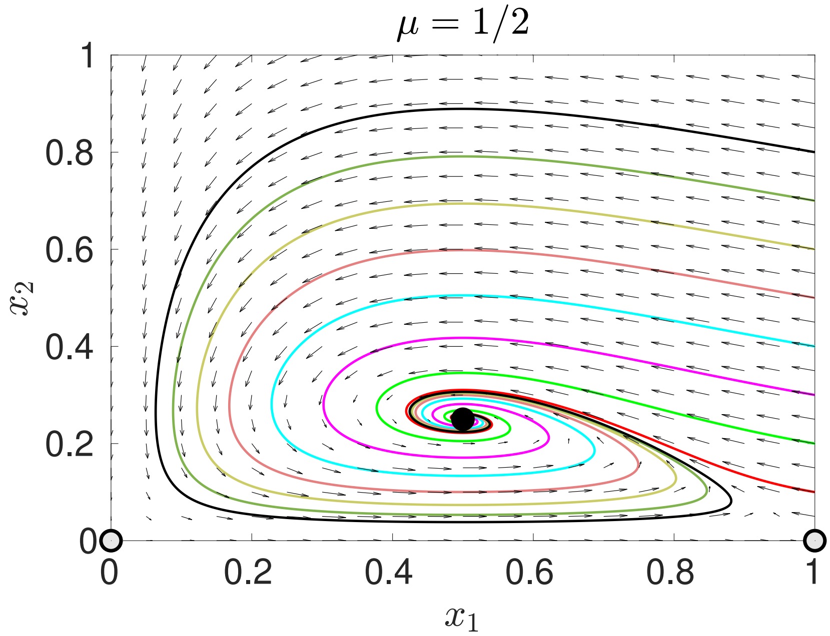


We can also draw the bifurcation diagram in the μ–x_1–x_2 space, when we can also add the (stable) limit cycles for $\mu < 1/3$. This plot (from two different viewing angles) is visualized below:

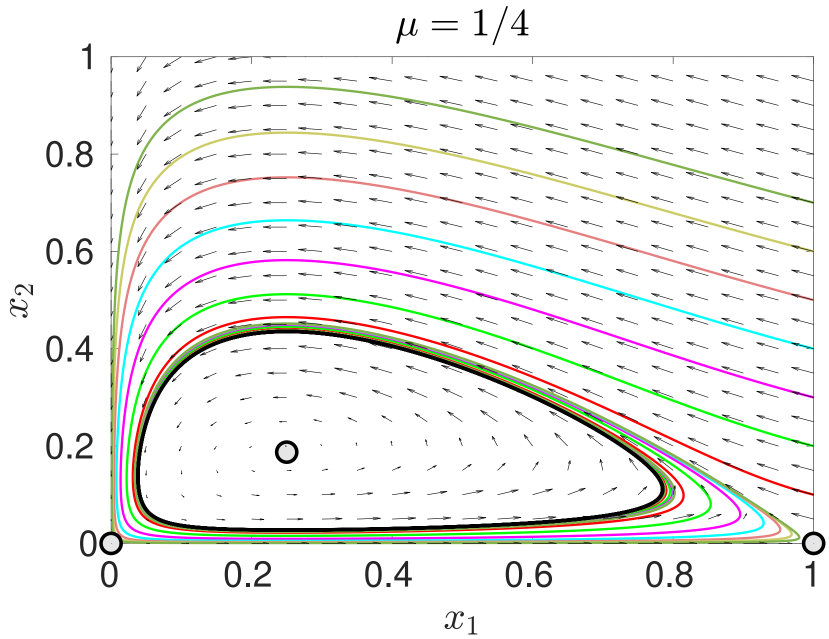




(b) If $\mu = 1/2$, then the fixed point $\mathbf{x}_{c1} = [\mu, \mu - \mu^2] = [1/2, 1/4]$ is a stable spiral and trajectories approach this stable critical point as shown below, where we plot eight trajectories starting at the right boundary of the square $[0,1] \times [0,1]$:



If $\mu = 1/4$, then the fixed point $\mathbf{x}_{c1} = [\mu, \mu - \mu^2] = [1/4, 3/16]$ is unstable and the system has a limit cycle as illustrated below, where we observe that trajectories (starting at the right boundary of the square $[0,1] \times [0,1]$) approach the limit cycle which is plotted using the black solid line:



Radek Erban: erban@maths.ox.ac.uk

2. Consider the system of $n = 2$ chemical species X_1 and X_2 which are subject to the following $\ell = 4$ chemical reactions:

$$X_1 \xrightarrow{k_1} X_2 \qquad \emptyset \xrightarrow{k_2} X_1 \qquad X_1 \xrightarrow{k_3} \emptyset \qquad 2X_1 + X_2 \xrightarrow{k_4} 3X_1$$

Let $x_1(t)$ and $x_2(t)$ be the concentrations of the chemical species X_1 and X_2, respectively.

(a) Assuming mass action kinetics, write a system of ODEs (reaction rate equations) describing the time evolution of $x_1(t)$ and $x_2(t)$.

(b) Assume the problem has already been non-dimensionalized and choose the values of dimensionless rate constants as

$$k_1 = \mu \qquad \text{and} \qquad k_2 = k_3 = k_4 = 1\,,$$

where $\mu > 0$ is a single parameter that we will vary.

Show that a supercritical Hopf bifurcation occurs at some parameter value $\mu = \mu_c$, where you should determine the value of $\mu = \mu_c$ at the bifurcation point.

(c) Find an approximation of the amplitude and the period of the limit cycle close to the bifurcation value $\mu = \mu_c$.

(d) Sketch the phase plane for μ close to $\mu = \mu_c$.

Solution:

(a) Using the definition of mass action kinetics (covered in Lecture 1), we have:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= k_2 - (k_1 + k_3)\, x_1 + k_4\, x_1^2\, x_2 \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= k_1\, x_1 - k_4\, x_1^2\, x_2
\end{aligned}$$

(b) Using our values of parameters $k_1 = \mu$, $k_2 = k_3 = k_4 = 1$, we have

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 1 - (\mu + 1)\, x_1 + x_1^2\, x_2 \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= \mu\, x_1 - x_1^2\, x_2
\end{aligned}$$

This system only has one critical point

$$\mathbf{x}_c = [1, \mu].$$

The Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 2x_1x_2 - \mu - 1 & x_1^2 \\ \mu - 2x_1x_2 & -x_1^2 \end{pmatrix},$$

giving

$$D\mathbf{f}(\mathbf{x}_c) = \begin{pmatrix} \mu - 1 & 1 \\ -\mu & -1 \end{pmatrix}.$$

The eigenvalues solve the quadratic equation

$$\lambda^2 + (2-\mu)\,\lambda + 1 = 0\,,$$

which implies

$$\lambda_\pm = \frac{\mu - 2 \pm \sqrt{\mu(\mu-4)}}{2}.$$

If $\mu \in (0,4)$, then we have two complex conjugate eigenvalues

$$\lambda_\pm = \frac{\mu-2}{2} \pm i\,\frac{\sqrt{\mu(4-\mu)}}{2}\,.$$

The real part is positive for $\mu > 2$ and negative for $\mu < 2$. We have a pair of purely imaginary eigenvalues at the bifurcation point $\mu = 2$, when

$$\lambda_\pm = \pm\, i$$

and the stability of the critical point $\mathbf{x}_c$ changes at the bifurcation point $\mu = 2$. Introducing new variables

$$\overline{x}_1 = x_1 - 1, \qquad \overline{x}_2 = x_2 - \mu\,, \qquad \overline{\mu} = \frac{\mu-2}{2}\,,$$

the ODE system transforms to

$$\begin{aligned} \frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} &= (2\,\overline{\mu}+1)\,\overline{x_1} + \overline{x_2} + 2\,\overline{x}_1\,\overline{x}_2 + 2\,(\,\overline{\mu}+1)\,\overline{x}_1^2 + \overline{x}_1^2\,\overline{x}_2 \\ \frac{\mathrm{d}\overline{x}_2}{\mathrm{d}t} &= -\,2\,(\overline{\mu}+1)\,\overline{x}_1 - \overline{x}_2 - 2\,\overline{x}_1\,\overline{x}_2 - 2\,(\overline{\mu}+1)\,\overline{x}_1^2 - \overline{x}_1^2\,\overline{x}_2 \end{aligned}$$

which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = M(\overline{\mu})\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} + \left(2\,\overline{x}_1\,\overline{x}_2 + 2\,(\overline{\mu}+1)\,\overline{x}_1^2 + \overline{x}_1^2\,\overline{x}_2\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad (2)$$

where matrix $M(\overline{\mu})$ is

$$M(\overline{\mu}) = \begin{pmatrix} 2\,\overline{\mu} + 1 & 1 \\ -\,2\,(\overline{\mu}+1) & -1 \end{pmatrix}$$

Close to the bifurcation point $\overline{\mu} = 0$, matrix $M(\overline{\mu})$ has eigenvalues

$$\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \,\pm\, i\,\omega(\overline{\mu})$$

where

$$\alpha(\overline{\mu}) = \overline{\mu} \qquad \text{and} \qquad \omega(\overline{\mu}) = \sqrt{1-\overline{\mu}^2}$$

which implies

$$\alpha(0) = 0\,, \qquad \omega(0) = 1\,, \qquad \alpha'(0) = 1\,, \qquad \text{and} \qquad \omega'(0) = 0\,.$$

Matrix $M(0)$ has eigenvalues $\lambda_\pm = \pm i$ with eigenvectors

$$\mathbf{v}_\pm = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \mp i \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We introduce change of variables

$$\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{with inverse} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix}.$$

Using (2) at the bifurcation point $\overline{\mu} = 0$, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} M(0) \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
&\quad + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \left(2\,\overline{x}_1\,\overline{x}_2 \,+\, 2\,\overline{x}_1^2 \,+\, \overline{x}_1^2\,\overline{x}_2\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \left(y_2^2 - y_1^2 \,+\, y_1^3 \,-\, 2\,y_1^2\,y_2 \,+\, y_1\,y_2^2\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix},
\end{aligned}$$

which is in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(y_1,y_2) \\ h_2(y_1,y_2) \end{pmatrix}$$

where

$$h_2(y_1,y_2) = -h_1(y_1,y_2) \,= y_2^2 - y_1^2 \,+\, y_1^3 \,-\, 2\,y_1^2\,y_2 \,+\, y_1\,y_2^2\,.$$

Evaluating the partial derivatives at the origin $\mathbf{0} = [0,0]$, we get

$$\begin{aligned}
a(0) &= \frac{1}{16}\left(\frac{\partial^3 h_1}{\partial y_1^3} + \frac{\partial^3 h_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 h_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 h_2}{\partial y_2^3}\right) + \frac{1}{16\,\omega(0)}\left[\frac{\partial^2 h_1}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_1}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\right)\right. \\
&\qquad \left. - \frac{\partial^2 h_2}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_2}{\partial y_2^2}\right) - \frac{\partial^2 h_1}{\partial y_1^2}\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\frac{\partial^2 h_2}{\partial y_2^2}\right] \\
&= \frac{1}{16}(-\,6 \,-\, 2 \,-\, 4 \,+\, 0) + \frac{1}{16\,\omega(0)}[0(2-2) - 0(-2+2) - 2(-2) - 2(2)] \\
&= -\,\frac{3}{4}\,.
\end{aligned}$$

Since $a(0) < 0$, we have a supercritical Hopf bifurcation at $\overline{\mu} = 0$, *i.e.* the original system has a supercritical Hopf bifurcation at $\mu = 2$.

Radek Erban: erban@maths.ox.ac.uk

(c) The normal form is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \overline{\mu}\, r - \frac{3}{4} r^3 + \dots$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + \dots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0$, *i.e* for $\mu < 2$ and unstable for $\overline{\mu} > 0$, *i.e* for $\mu > 2$. A stable limit cycle is born with amplitude

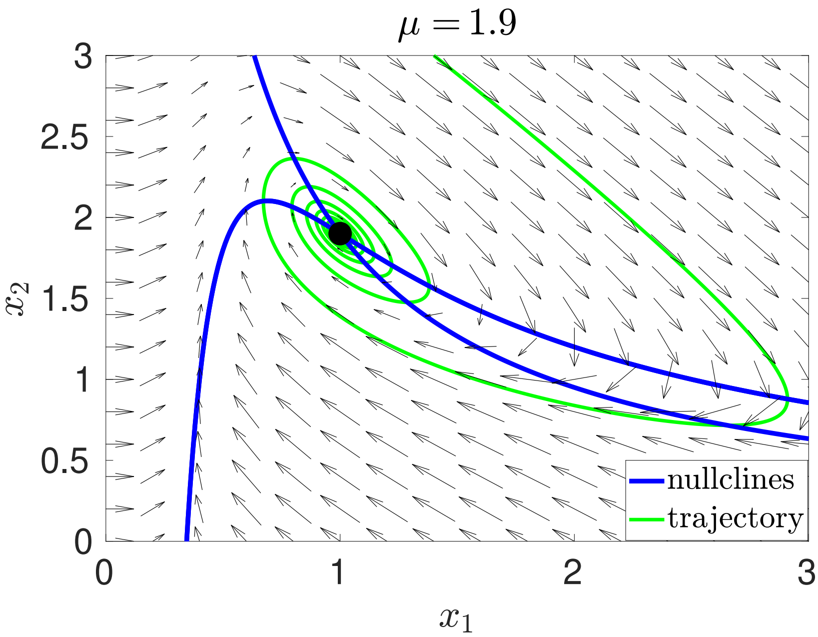
$$\sqrt{\frac{2(\mu-2)}{3}}$$

and period 2π at the bifurcation point for $\mu > 2$. The limit cycle can be approximated by

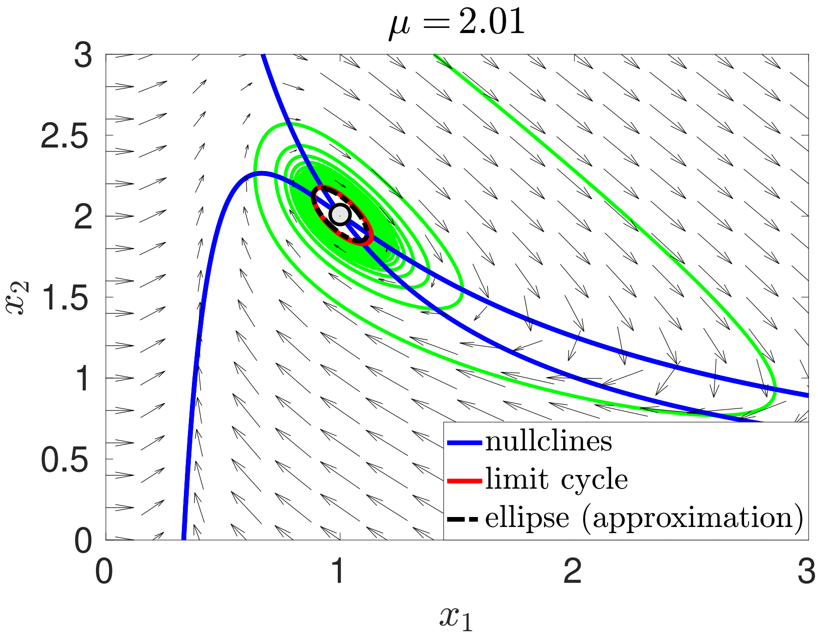
$$y_1^2 + y_2^2 = \frac{2(\mu-2)}{3}$$

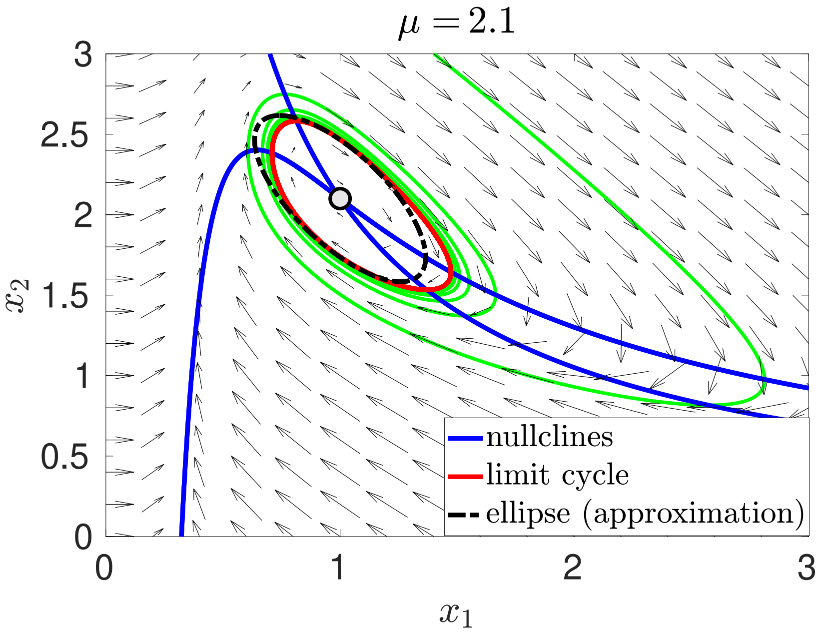
which corresponds to an ellipse in x_1 and x_2 variables.

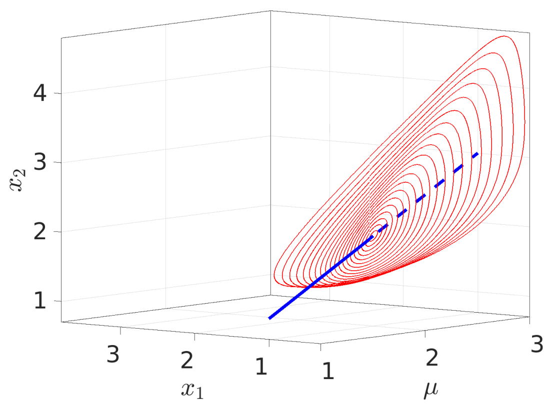
(d) If $\mu < 2$, then the fixed point $\mathbf{x}_c = [1, \mu]$ is a stable spiral and trajectories approach this stable critical point as shown below for $\mu = 1.9$, where we plot a trajectory starting at $[1.4, 3]$ as the green line. Nullclines are visualized as blue lines:

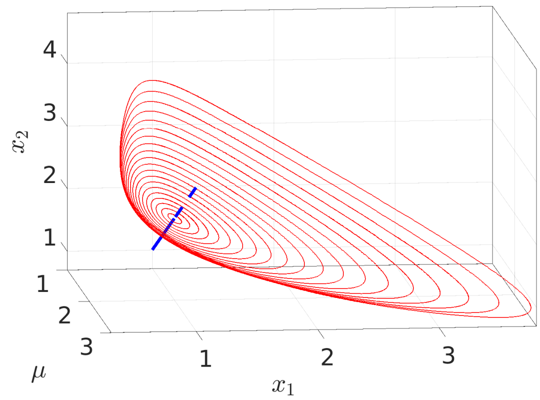


If $\mu > 2$, then the fixed point $\mathbf{x}_c = [1, \mu]$ is unstable and the system has a limit cycle as illustrated on the next page, where we observe that the green trajectory approaches the red limit cycle. The approximating ellipse is visualized as the black dashed line. The bifurcation diagram can also be visualized in the μ–x_1–x_2 space, as shown on the next page, with (stable) limit cycles included for $\mu > 2$.









Radek Erban: erban@maths.ox.ac.uk

3. Consider the second-order ODE describing an 'asymmetric spring' in the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\,x \,+\, \varepsilon\, x^2$$

(a) Rewrite the ODE as a planar system of autonomous ODEs.

(b) Find and classify all critical points.

(c) Consider the periodic orbit satisfying

$$x(0) = A, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0.$$

Use the Poincaré-Lindstedt method to find the expansion of the frequency of this orbit up to [and including] terms of $\mathcal{O}(\varepsilon^2)$.

Solution:

(a) Denoting

$$y_1 = x \qquad \text{and} \qquad y_2 = \frac{\mathrm{d}x}{\mathrm{d}t}\,,$$

we can rewrite this second order equation as the following planar system of autonomous ODEs

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_2 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= -\,y_1 \,+\, \varepsilon\, y_1^2 \end{aligned}$$

(b) The critical points are obtained by solving $0 = -\,y_1 \,+\, \varepsilon\, y_1^2$ and $y_2 = 0$. We get $\mathbf{y}_{c1} = [0, 0]$ which exists for all $\varepsilon \in \mathbb{R}$ and $\mathbf{y}_{c2} = [\varepsilon^{-1}, 0]$ which exists for $\varepsilon \neq 0$. The Jacobian matrix is

$$D\mathbf{f}(\mathbf{y}) = \begin{pmatrix} 0 & 1 \\ -1 \,+\, 2\,\varepsilon\, y_1 & 0 \end{pmatrix},$$

giving

$$D\mathbf{f}(\mathbf{y}_{c1}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad D\mathbf{f}(\mathbf{y}_{c2}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_\pm = \pm\, i$ at $\mathbf{y}_{c1}$ and $\lambda_\pm = \pm 1$ at $\mathbf{y}_{c2}$. Consequently, $\mathbf{y}_{c1}$ is a center and $\mathbf{y}_{c2}$ is a saddle whenever it exists.

(c) We transform the time variable as $\tau = \omega(\varepsilon)\, t$ where $2\pi/\omega(\varepsilon)$ is the period of the periodic solution. We obtain

$$\omega^2(\varepsilon)\, \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = -\,x \,+\, \varepsilon\, x^2 \tag{3}$$

Radek Erban: erban@maths.ox.ac.uk

where we denote the solution by $x(\tau;\varepsilon)$. We have $x(\tau+2\pi;\varepsilon)=x(\tau;\varepsilon)$ and, by translating time if necessary, we also have

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}(0;\varepsilon)=0 \qquad \text{and} \qquad x(0;\varepsilon)=A\,,$$

where A is the amplitude. We expand

$$x(\tau;\varepsilon)=x_0(\tau)\,+\,\varepsilon\,x_1(\tau)\,+\,\varepsilon^2\,x_2(\tau)\,+\,\dots \;\text{ and }\; \omega(\varepsilon)=\omega_0\,+\,\varepsilon\,\omega_1\,+\,\varepsilon^2\,\omega_2\,+\,\dots$$

Substituting into (3) and equating coefficients of ε^0 gives

$$\omega_0^2\,\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2}\;=\;-\,x_0 \qquad \text{with} \qquad x_0(\tau+2\pi)=x_0(\tau)\,.$$

Thus $\omega_0=1$ and

$$x_0(\tau)=A\,\cos(\tau)\,.$$

Equating coefficients of ε^1 gives

$$\omega_0^2\,\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2}\,+\,2\,\omega_0\,\omega_1\,\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2}\;=\;-\,x_1\,+\,x_0^2 \qquad \text{with} \qquad x_1(\tau+2\pi)=x_1(\tau)\,.$$

Substituting $\omega_0=1$ and $x_0(\tau)=A\,\cos(\tau)$, we get

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2}\,+\,x_1\;=\;2\,\omega_1\,A\cos(\tau)\;+\;A^2\cos^2(\tau)\;=\;2\,\omega_1\,A\cos(\tau)\;+\;\frac{A^2}{2}\,(1+\cos(2\tau))\,.$$

Eliminating the secular term gives $\omega_1=0$. Solving the resulting equation, we get

$$x_1(\tau)=c_1\cos(\tau)+c_2\sin(\tau)+\frac{A^2}{2}-\frac{A^2}{6}\cos(2\tau)\,. \tag{4}$$

Using the initial conditions, we get $c_1=-A^2/3$ and $c_2=0$. Consequently, we have

$$x(\tau;\varepsilon)=A\,\cos(\tau)+\varepsilon\left(\frac{A^2}{2}-\frac{A^2}{3}\cos(\tau)-\frac{A^2}{6}\cos(2\tau)\right)+\mathcal{O}(\varepsilon^2).$$

Equating coefficients of ε^2 gives

$$\omega_0^2\,\frac{\mathrm{d}^2x_2}{\mathrm{d}\tau^2}\,+\,2\,\omega_0\,\omega_1\,\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2}\,+\,(2\,\omega_0\,\omega_2\,+\,\omega_1^2)\,\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2}\;=\;-\,x_2\,+\,2\,x_0\,x_1\,.$$

Substituting $\omega_0=1$, $\omega_1=0$, $x_0(\tau)=A\,\cos(\tau)$ and equation (4) for $x_1(\tau)$, we get

$$\begin{aligned}\frac{\mathrm{d}^2x_2}{\mathrm{d}\tau^2}\,+\,x_2 \;&=\; 2\,\omega_2\,A\,\cos(\tau)\,+\,2\,A\,\cos(\tau)\left(\frac{A^2}{2}-\frac{A^2}{3}\cos(\tau)-\frac{A^2}{6}\cos(2\tau)\right)\\ &=\;\left(2\,\omega_2\,A\,+\,\frac{5A^3}{6}\right)\cos(\tau)\;-\;\frac{A^3}{6}\,\left(2+2\cos(2\tau)+\cos(3\tau)\right).\end{aligned}$$

Eliminating the secular term gives

$$2\,\omega_2\,A\,+\,\frac{5A^3}{6}=0$$

which implies

$$\omega_2=-\frac{5A^2}{12}.$$

Thus, we conclude that

$$\omega=1-\frac{5\,\varepsilon^2A^2}{12}+\mathcal{O}\big(\varepsilon^3\big)\,.$$

Radek Erban: erban@maths.ox.ac.uk

Section C: Problem 7

7. Hilbert's 16th problem considers planar ODEs which can be written as

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= f_1(x_1,x_2)\,, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= f_2(x_1,x_2)\,,\end{aligned} \qquad (*)$$

where $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ are real polynomials of degree at most n. In its classic formulation, Hilbert's 16th problem asks questions about the number and positions of limit cycles that such a system of ODEs can have. Denoting $H(n)$ the maximum number of limit cycles for such ODE systems, neither the value of $H(n)$ (for $n \geq 2$) nor any upper bound on $H(n)$ have yet been found.

(a) Show that $H(2) \geq 1$ by finding quadratic polynomials $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ such that the ODE system $(*)$ has a limit cycle.

(b) Show that $H(n) \geq 2$ by finding polynomials $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ of degree n such that the ODE system $(*)$ has two limit cycles. Choose the degree n of the polynomials in your example as small as you can.

(c) Show that $H(n) \geq 4$ by finding polynomials $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ of degree n such that the ODE system $(*)$ has four limit cycles. Choose the degree n of the polynomials in your example as small as you can.

(d) Some ODEs in the form $(*)$ describe chemical reaction networks with two chemical species (for example, we have seen examples of such ODEs in Questions 2 and 5 on this Problem Sheet). Are your answers to parts (b) and (c) describing some chemical systems, *i.e.* could you find a chemical reaction network described by your ODEs? If not, can you solve parts (b) and (c) to find ODEs that describe chemical reaction systems and have the corresponding number of limit cycles?

Solution:

(a) An example was presented in our Lecture 11, when we discussed the homoclinic (saddle-loop) bifurcation. It was given as follows:

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -\, x_1\end{aligned}$$

To analyze this system, we observed that there are two critical points at

$$\mathbf{x}_{c1} = [0,0] \qquad \text{and} \qquad \mathbf{x}_{c2} = [0,1].$$

The Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \mu - x_2 & 1 - 2x_2 - x_1 \\ -1 & 0 \end{pmatrix},$$

giving

$$D\mathbf{f}(\mathbf{x}_{c1}) = \begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad D\mathbf{f}(\mathbf{x}_{c2}) = \begin{pmatrix} \mu - 1 & -1 \\ -1 & 0 \end{pmatrix}.$$

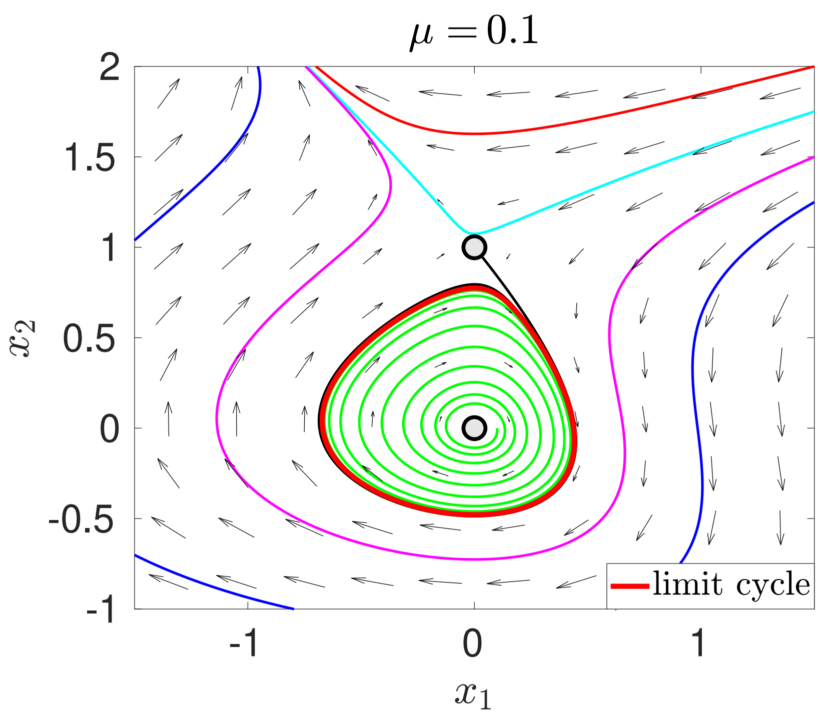
We considered the eigenvalues of $D\mathbf{f}(\mathbf{x}_{c1})$, *i.e.*

$$\lambda_{\pm} = \frac{\mu}{2} \pm \frac{\sqrt{\mu^2 - 4}}{2}$$

to deduce that there is a supercritical Hopf bifurcation at $\mu = 0$. The limit cycle then exists in interval

$$\mu \in (0, 0.135454802155\dots).$$

It is created at $\mu = 0$ using the supercritical Hopf bifurcation and it disappears at $\mu = 0.135454802155\dots$ using the homoclinic (saddle-loop) bifurcation. Consequently, we can choose $\mu = 0.1$ to get an example showing that $H(2) \geq 1$. The phase plane is plotted here:



(b) There are different constructions which can lead to planar polynomial systems with two limit cycles. For example, you can start in polar coordinates with

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(r^2-1)(r^2-4) \qquad \text{and} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 \tag{5}$$

which has two limit cycles with radius $r = 1$ and $r = 2$ and transform it to the ODE system $(*)$ with $n = 5$ given by

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_2 + x_1(x_1^2+x_2^2-1)(x_1^2+x_2^2-4) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 + x_2(x_1^2+x_2^2-1)(x_1^2+x_2^2-4) \end{aligned}$$

The value $n = 5$ is not optimal and it can be decreased to $n = 2$, as discussed in Lecture 13.

(c) As in part (b), we could start in polar coordinates where system (5) would be replaced by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(r^2-1)(r^2-4)(r^2-9)(r^2-16) \qquad \text{and} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$$

giving us a polynomial system with four limit cycles (at $r = 1$, $r = 2$, $r = 3$ and $r = 4$) which transforms to the ODE system $(*)$ with $n = 9$. As discussed in Lecture 13, you could construct systems with smaller values of n. In fact, quadratic systems with four limit cycles have been found in the literature, so the optimal answer to this question would again be $n = 2$.

(d) The polynomial systems discussed in parts (a), (b) and (c) are not representing any chemical systems. To construct chemical systems, we could first shift the dynamics to positive quadrant (by translation) and then multiply the right hand side of the first ODE in $(*)$ by x_1 and the second ODE in $(*)$ by x_2, which would increase the degree n by one. In fact, one can prove that chemical systems require at least one cubic nonlinearity on the right hand side (all chemical systems with limit cycles in our problem sheets and lectures included a trimolecular reaction leading to such a cubic nonlinearity). For further discussion and constructions of chemical systems with limit cycles, see:

R. Erban and H.W. Kang, "Chemical Systems with Limit Cycles", *Bulletin of Mathematical Biology* **85**, article 76 (2023)

Radek Erban: erban@maths.ox.ac.uk