Geometric Group Theory

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Amalgams

We will describe the elements of an amalgamated product $A *_H B$ by words.

Simplified notation: we identify H with $\alpha(H)$ and $\beta(H)$, and we identify A with $i_A(A)$, B with $i_B(B)$.

Choose A_1, a set of right coset representatives of H in A, and B_1 a set of right coset representatives of H in B, such that $1 \in A_1$, $1 \in B_1$.

Definition

A reduced word of the amalgam $A *_H B$ is a word of the form $(h, s_1, \ldots, s_n)$, $h \in H$, $s_i \in A_1 \cup B_1$, $s_i \neq 1$, s_i alternating from A_1 to B_1. We associate to this the element $hs_1 \ldots s_n$ of $A *_H B$. The length of the reduced word is n.

Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

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Proof: For all $g \in G$, we can write $g = a_1 b_1 ... a_m b_m$ for some $a_i \in A$, $b_i \in B$.

We claim that g can be represented by a reduced word $(h, s_1, ..., s_n)$.

$m = 1$: $g = a_1 b_1 = a_1 \bar{h} b' = \underbrace{a_1 h}_{\in A} b' = h' a' b'$, where $a' \in A_1$, $b' \in B_1$.

Inductive step: exercise.

Uniqueness: Let X be the set of all reduced words. We will define an action of G on X, i.e. a group homomorphism

$$G \to Symm(X) = Bij(X)$$

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By the universal property, it suffices to define $\alpha_1 : A \to Symm(X)$, $\beta_1 : B \to Symm(X)$ such that $\alpha_1(h) = \beta_1(\bar{h})$.

Definition of α_1: Consider $a \in A$.

Case 1: $a = h_0 \in H$:

$$h_0 \cdot (h, s_1, ..., s_n) = (h_0 h, s_1, ..., s_n)$$

Case 2: $a \in A \setminus H$.

2.a: $s_1 \in B$. $\forall h \in H$, write $ah = h'a'$ where $a' \in A_1$, $a' \neq 1$.

$$a \cdot (h, s_1, ..., s_n) = (h', a', s_1, ..., s_n)$$

2.b: $s_1 \in A$, $s_2 \in B$. $\forall h \in H$, write $ahs_1 = h'a'$, $a' \in A_1$.

$$\begin{aligned} a \cdot (h, s_1, ..., s_n) &= (h', a', s_2, ..., s_n) \quad \text{if } a' \neq 1 \\ &= (h', s_2, ..., s_n) \quad \text{if } a' = 1 \end{aligned}$$

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This defines a map $\sigma_a : X \to X$.

Exercise: Check that $\sigma_{a_1 a_2} = \sigma_{a_1} \circ \sigma_{a_2}$.

Therefore $\sigma_a \circ \sigma_{a^{-1}} = \mathrm{id}$ and so σ_a is a bijection.

So we have defined $\alpha_1 : A \to Symm(X), \alpha_1(a) = \sigma_a$.

Likewise, we can define $\beta_1 : B \to Symm(X)$.

We have that $\alpha_1(h) = \beta_1(h) = \sigma_h$, for every $h \in H$.

Therefore there exists a unique $\varphi : A *_H B \to Symm(X)$.

Exercise: $\forall g \in G$, if $g = hs_1 ... s_n$, a reduced word, then

$$\varphi(g)(1) = (h, s_1, ..., s_n).$$

Thus, the reduced word is unique. □

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Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

Corollary

i_A and i_B are injective. Hence A, B can be seen as subgroups of $A *_H B$.

Corollary

If $(g_1, ..., g_n)$, $n \geq 2$, is such that $g_i \in A \cup B$, $g_i \notin H$, $\forall i \geq 2$, and g_i alternate between A and B, then $g_1 ... g_n \neq 1$ in $A *_H B$.

Proof.

Use induction to show that it can be represented by a reduced word of length $n - 1$ if $g_1 \in H$ or of length n if $g_1 \notin H$. □

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Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

Corollary

In G, $A \cap B = H$.

Definition

The reduced word $(h, s_1, ..., s_n)$ and the reduced element $hs_1...s_n \in A *_H B$ are cyclically reduced if $n \geq 2$ and $s_1 s_n$ is reduced.

Proposition

- *Every $g \in A *_H B$ is conjugate either to a cyclically reduced element or to some $a \in A$ or to some $b \in B$.*
- *Every cyclically reduced element has infinite order.*

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Proposition

1. *Every $g \in A *_H B$ is conjugate either to a cyclically reduced element or to some $a \in A$ or to some $b \in B$.*
2. *Every cyclically reduced word has infinite order.*

Proof: (1) : If $g = hs_1 \ldots s_n$ is not cyclically reduced, i.e. s_1, s_n are both in A or both in B, then $s_n g s_n^{-1}$ is represented by a word of length $n-1$. Repeat until we have a cyclically reduced word or a word of length 1.

(2) : If g is cyclically reduced of length n then g^k has length kn, so $g^k \neq 1$. □

Corollary

Given any finite subgroup $F \leq A *_H B$, F must be contained in a conjugate gAg^{-1} or gBg^{-1}.

Proof: exercise.

The unique root property

Proposition

Every $u \in F(X)$ is conjugate to a cyclically reduced word.

Corollary (unique root property)

If $g, h \in F(X)$ are such that $g^k = h^k$ for some k then $g = h$.

Question: Find G torsion-free group s.t. $\exists g \neq h$ with $g^k = h^k$ for some k.
Take $G = \langle g, h \mid g^k = h^k \rangle$. It is an amalgamated product $G = A *_H B$, where $A = \langle g \rangle$, $B = \langle h \rangle$, and $H = \mathbb{Z} \simeq \langle g^k \rangle \simeq \langle h^k \rangle$.
Exercise: If every pair of distinct elements have an equal power then $G = \mathrm{Tor}\, G$. NB This does not mean that G is finite. See for instance https://en.wikipedia.org/wiki/Burnside_problem
Example due to Olshanskii: There exist finitely generated, non-cyclic, torsion-free groups G where any two elements have equal powers, i.e., for any g, h there exist m, n such that $g^m = h^n$.

Amalgams and actions on trees

Definition

- Suppose G is a group acting on a graph X. We say that G acts on X without inversions if for every $g \in G$ and $[v, w] \in E(X)$ we have that $g([v, w]) \neq [w, v]$.
- A free action of G on X is an action that is free on the vertices and without inversions.

Suppose G is a group acting freely on a tree T.

A subtree $S \subseteq T$ is a fundamental domain if it intersects the orbit $G \cdot v$ of every vertex v of T, and it intersects the orbit of every edge exactly once.

Theorem

$G = A *_H B$ acts on a tree T with fundamental domain an edge $[P, Q]$ such that $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.