C3.11 Riemannian Geometry

Sheet 2 — HT24

Solutions

This problem sheet is based on Sections 2.3 and 3 of the lecture notes. This version contains the solutions to all of the questions.

Section A

1. Let (M, g) be an oriented Riemannian manifold. Show that parallel transport along any curve in M is orientation preserving.

   Solution: Let $\alpha : [0, L] \to M$ be a curve and let $\tau_t : T_{\alpha(0)} \to T_{\alpha(t)}M$ be the parallel transport along $\alpha|_{[0,t]}$ for each $t \in [0, L]$. Then $\tau_1 = \tau_\alpha$ is the parallel transport along α.

   Choose an oriented orthonormal basis $\{E_1(0), \ldots, E_n(0)\}$ for $T_{\alpha(0)}M$. Since τ_t is an isometry for all t, if we define $E_i(t) = \tau_t(E_i(0))$, then $\{E_1(t), \ldots, E_n(t)\}$ is an orthonormal basis for $T_{\alpha(t)}M$.

   If we use the basis $\{E_1(t), \ldots, E_n(t)\}$ as the basis for $T_{\alpha(t)}M$, the map $f : [0, L] \to \mathbb{R}$ given by
   $$f(t) = \det(\tau_t)$$
   is then well-defined and continuous (in fact, smooth). Moreover, $f(t) \in \{\pm 1\}$ for all t since $\{E_1(t), \ldots, E_n(t)\}$ is orthonormal for all t.

   As $[0, L]$ is connected and $f(0) = 1$, we deduce that $f(t) = 1$ for all t. Hence τ_t is orientation preserving for all t, and so $\tau_\alpha = \tau_1$ is orientation preserving.

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2. Let (M, g) be an n-dimensional Riemannian manifold. Let $p \in M$ and let U be a normal neighbourhood of p.

   Let $\{E_1, \ldots, E_n\}$ be an orthonormal basis for T_pM, let $\psi : T_pM \to \mathbb{R}^n$ be given by $\psi(\sum_{i=1}^n x_i E_i) = (x_1, \ldots, x_n)$ and let $\varphi = \psi \circ \exp_p^{-1} : U \to \mathbb{R}^n$.

   (a) Let $\gamma(t)$ be a geodesic through p in U in M. Show that

   $$\varphi \circ \gamma(t) = (a_1 t, \ldots, a_n t)$$

   for $(a_1, \ldots, a_n) \in \mathbb{R}^n$.

   (b) Show that in (U, φ), we have $g_{ij}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$.

   (c) Hence, or otherwise, show that there is open set $V \ni p$ and orthonormal vector fields $E_1, \ldots, E_n$ on V such that

   $$\nabla_{E_i} E_j(p) = 0.$$

Solution:

(a) Geodesics through p in U are given by $\gamma(t) = \exp_p(tX)$ for some X in T_pM since U is a normal neighbourhood (which means $\exp : V \to U$ is a diffeomorphism for some $V \subseteq T_pM$). Thus

$$\varphi \circ \gamma(t) = \psi \circ \exp_p^{-1}(\exp_p(tX)) = \psi(tX).$$

As $\{E_1, \ldots, E_n\}$ is a basis, we can write any $X \in T_pM$ uniquely as $X = \sum_{i=1}^n a_i E_i$ for some $(a_1, \ldots, a_n) \in \mathbb{R}^n$. Hence,

$$\varphi \circ \gamma(t) = \psi(tX) = \psi(\sum_{i=1}^n t a_i E_i) = (a_1 t, \ldots, a_n t)$$

by definition, as we wanted to show.

(b) We see that $\varphi(p) = 0$ and $\psi(E_i) = \mathbf{e}_i$ (the unit vector with 1 in the ith place and 0 everywhere else). Therefore, $\mathrm{d}\psi_0(E_i) = \partial_i$. Hence, taking the inverse (as ψ is a local diffeomorphism at 0, its differential at 0 is invertible), we have $\mathrm{d}(\psi^{-1})_0(\partial_i) = E_i$.

Since $\varphi = \psi \circ \exp_p^{-1}$ we have $\varphi^{-1} = \exp_p \circ \psi^{-1}$ so, by the Chain rule, we have

$$\mathrm{d}(\varphi^{-1})_{\varphi(p)}(\partial_i) = \mathrm{d}(\exp_p)_0 \circ \mathrm{d}(\psi^{-1})_0(\partial_i) = \mathrm{d}(\exp_p)_0(E_i) = E_i$$

since $\mathrm{d}(\exp_p)_0 = \mathrm{id}$. Thus

$$g_{ij}(p) = g_p(\mathrm{d}(\varphi^{-1})_{\varphi(p)}(\partial_i), \mathrm{d}(\varphi^{-1})_{\varphi(p)}(\partial_j)) = g_p(E_i, E_j) = \delta_{ij}.$$

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Finally, the geodesic equations in U for $\gamma(t) = (x_1(t), \ldots, x_n(t))$ are

$$x_k''(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^k x_i'(t) x_j'(t) = 0.$$

However, as we know that geodesics through p are given by straight lines $x_i(t) = a_i t$, we see that $x_k'' = 0$ and $x_i'(t) = a_i$, therefore

$$\sum_{i,j=1}^{n} \Gamma_{ij}^k(p) a_i a_j = 0$$

for all $(a_1, \ldots, a_n) \in \mathbb{R}^n$. We deduce that $\Gamma_{ij}^k(p) = 0$.

[Note: The chart (U, φ) is called *geodesic normal coordinates* at p. These coordinates are very useful for calculations and say that there are no zero or first order invariants of a Riemannian metric at a given point.]

(c) Take V to be a geodesic ball $B_\epsilon(p)$ contained in U. For every point q in V there exists a unique radial geodesic γ_q from p to q. Let $\tau_q : T_pM \to T_qM$ be the parallel transport along γ_q. We then define, for all $q \in V$,

$$E_i(q) = \tau_q(E_i(p))$$

where $\{E_1(p), \ldots, E_n(p)\}$ is an orthonormal basis for T_pM.

Since τ_q is an isometry for all q and depends smoothly on q, we deduce that $E_1, \ldots, E_n$ are orthonormal vector fields on V.

Since the integral curve α of E_i with $\alpha(0) = p$ is a radial geodesic, and E_j is parallel along radial geodesics, so $\tau_t^{-1}(E_j(\alpha(t)) = E_j(p)$ for all t (in the notation of Question 1) we deduce from Question 1 that

$$\nabla_{E_i} E_j(p) = 0.$$

[Note: The frame $\{E_1, \ldots, E_n\}$ is a *geodesic frame* on V or near p. This is very useful for computations.]

Section B

3. Let X, Y be vector fields on (M, g). Let $p \in M$ and let $\alpha : (-\epsilon, \epsilon) \to M$ be the integral curve of X with $\alpha(0) = p$. For all $t \in (-\epsilon, \epsilon)$ let $\tau_t : T_pM \to T_{\alpha(t)}M$ be parallel transport along $\alpha|_{[0,t]}$. Show that

$$\nabla_X Y(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tau_t^{-1}\big(Y(\alpha(t))\big)\Big)|_{t=0}.$$

Solution: Choose a chart (U, φ) with $p \in U$ so that $\varphi(p) = 0$ and let $\{X_1, \ldots, X_n\}$ be the coordinate frame field on U. Suppose, by making ϵ smaller if necessary, that $\alpha(-\epsilon, \epsilon) \subseteq U$.

Let $E_1(t), \ldots, E_n(t)$ be parallel vector fields along α so that $E_i(0) = X_i(p)$ for $i = 1, \ldots, n$, i.e. $E_i(t) = \tau_t(X_i(p))$ for all t. Since τ_t is an isomorphism for all t, we have that $\{E_1(t), \ldots, E_n(t)\}$ is a basis for $T_{\alpha(t)}M$ for all t.

Hence, we can write Y along α as

$$Y(\alpha(t)) = \sum_{i=1}^{n} b_i(t)E_i(t)$$

for some smooth functions $b_i : (-\epsilon, \epsilon) \to \mathbb{R}$. Since α is an integral curve of X, $X = \alpha'$ along α, and therefore

$$\nabla_X Y(p) = \nabla_{\alpha'} Y(p) = \nabla_{\alpha'} Y(\alpha(t))|_{t=0} = \nabla_{\alpha'} \sum_{i=1}^{n} b_i(t)E_i(t)|_{t=0} = \sum_{i=1}^{n} b_i'(0)E_i(0)$$

since $\nabla_{\alpha'}E_i = 0$.

On the other hand, we see that

$$\tau_t^{-1}\big(Y(\alpha(t))\big) = \tau_t^{-1}\left(\sum_{i=1}^{n} b_i(t)E_i(t)\right) = \sum_{i=1}^{n} b_i(t)E_i(0)$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\tau_t^{-1}\big(Y(\alpha(t))\big)\Big)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} b_i(t)E_i(0)\right)|_{t=0} = \sum_{i=1}^{n} b_i'(0)E_i(0) = \nabla_X Y(p)$$

as claimed.

[Note: This question says that you can recover the Levi-Civita connection from the parallel transport maps. Hence, parallel transport (which allows us to "connect" tangent spaces) is equivalent to the Levi-Civita connection.]

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4. Let (M,g) be a Riemannian manifold. Recall that a *Killing field* on M is a vector field X such that $\mathcal{L}_X g = 0$ or, equivalently, that the flow of X near any point consists of local isometries.

   (a) Let $p \in M$ and let U be a normal neighbourhood of p. Suppose that X is a Killing field on (M,g) so that $X(p) = 0$ and $X(q) \neq 0$ for all $q \in U \setminus \{p\}$.

   By using the First variation formula, or otherwise, show that X is tangent to all sufficiently small geodesic spheres centred at p.

   (b) Show that X is a Killing field on (M,g) if and only if, for all vector fields Y, Z on M,

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0.$$

Solution:

(a) By making U smaller if necessary we can assume that p is still an isolated zero of X in U and that the flow of X $\{\phi_t^X : U \to M \,:\, t \in (-\epsilon, \epsilon)\}$ is defined on U.

Let $\mathcal{S}_\delta(p)$ be any geodesic sphere contained in U and let $q \in \mathcal{S}_\delta(p)$. Let $\alpha : [0,\delta] \to U$ be the normalized radial geodesic from p to q. We define a variation $f : (-\epsilon,\epsilon) \times [0,\delta] \to M$ of α by

$$f(s,t) = \phi_s^X\big(\alpha(t)\big).$$

Notice that since $\phi_0^X = \mathrm{id}$, and the maps ϕ_s^X depend smoothly on s, f is indeed a variation of f. Moreover, the variation field V_f of f is given by

$$V_f(t) = \frac{\partial f}{\partial s}(0,t) = \frac{\mathrm{d}}{\mathrm{d}s}\phi_s^X|_{s=0}(\alpha(t)) = X(\alpha(t))$$

since ϕ_s^X is the flow of X. Hence,

$$V_f(0) = X(p) = 0 \quad \text{and} \quad V_f(\delta) = X(q).$$

We now notice that

$$|\frac{\partial f}{\partial t}(s,t)|^2 = |\mathrm{d}(\phi_s^X)_{\alpha(t)}(\alpha'(t))|^2 = |\alpha'(t)|^2$$

since X is Killing and so its flow consists of local isometries. Therefore, the energy of the variation f of α

$$E_f(s) = \int_0^\delta |\alpha'(t)|^2 \mathrm{d}t$$

is *independent of* s. (More geometrically, since X is Killing with $X(p) = 0$ it will map radial geodesics to radial geodesics and preserve their length, and hence their energy as they are geodesics.)

We deduce from the First variation formula that

$$\begin{aligned} 0 &= \frac{1}{2}E_f'(0) = -\int_0^\delta g(V_f, \nabla_{\alpha'}\alpha')\mathrm{d}t - g(V_f(0), \alpha'(0)) + g(V_f(\delta), \alpha'(\delta) \\ &= g(X(q), \alpha'(\delta)). \end{aligned}$$

Here we used that $\nabla_{\alpha'}\alpha' = 0$ as α is a geodesic and that $V_f(0) = 0$, $V_f(\delta) = X(q)$. Now, by the Gauss Lemma, as α is a normalized radial geodesic, we have that $\alpha'(\delta)$ is normal to $\mathcal{S}_\delta(p)$ at q, and is non-zero, and hence $X(q)$ must be tangent to $\mathcal{S}_\delta(p)$ at q.

Since q and δ are arbitrary, the result follows.

(b) We note that the given equation is called the *Killing equation*: for all Y, Z in M

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0.$$

The slick way to do the question is:

$$\begin{aligned} \big(\mathcal{L}_X g\big)(Y, Z) &= \mathcal{L}_X(g(Y, Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) \\ &= X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) + g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X). \end{aligned}$$

Using properties (iv)-(v) of the Levi-Civita connection (metric compatibility and torsion-free). Hence $\mathcal{L}_X g = 0$ (i.e. X is a Killing field) if and only if the Killing equation holds for all Y, Z in M.

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5. Let (M, g) be a Riemannian manifold, let $f : M \to \mathbb{R}$ be a smooth function and let X be a vector field on M.

   (a) Note that we have a linear map from vector fields to vector fields given by $Y \mapsto \nabla_Y X$. We define the *divergence* of X to be the smooth function

   $$\operatorname{div} X = \operatorname{tr}(Y \mapsto \nabla_Y X).$$

   Show that if X is a Killing field then $\operatorname{div} X = 0$.

   (b) Recall that $Y \mapsto g(Y, .)$ defines an isomorphism between vector fields and 1-forms on M. We define the *gradient* of f to be the vector field ∇f given by

   $$g(\nabla f, .) = \mathrm{d}f.$$

   We define the *Laplacian* of f to be the smooth function

   $$\Delta f = \operatorname{div} \nabla f.$$

   Show that

   $$\Delta(f^2) = 2f\Delta f + 2|\nabla f|^2.$$

   Now suppose further that M is compact, connected and oriented with Riemannian volume form Ω.

   (c) Show that

   $$\mathcal{L}_X \Omega = (\operatorname{div} X)\Omega.$$

   Relate this to the result about Killing fields from (a).

   (d) Show that if $\Delta f \geq 0$ on M then f is constant.

Solution:

(a) By the Killing equation (Question 4(b)), if X is Killing then

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$$

for all vector fields Y, Z, i.e. the map $Y \mapsto \nabla_Y X$ is skew-symmetric, so $\operatorname{div} X = 0$. Explicitly, if we choose a geodesic frame $\{E_1, \ldots, E_n\}$ on $V \ni p$ in (M, g) as in Question 3(c) then

$$0 = 2\sum_{i=1}^{n} g(\nabla_{E_i} X, E_i) = 2\operatorname{tr}(Y \mapsto \nabla_Y X) = 2\operatorname{div} X$$

on V. Since p is arbitrary, $\operatorname{div} X = 0$ everywhere.

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(b) We see from our solution to (a) that if we choose $p \in (M, g)$ and a geodesic frame $\{E_1, \ldots, E_n\}$ on $V \ni p$ then

$$\operatorname{div} X = \sum_{i=1}^{n} g(\nabla_{E_i} X, E_i).$$

We further see that

$$g(\nabla f, E_i) = \mathrm{d}f(E_i) = E_i(f)$$

and hence

$$\nabla f = \sum_{i=1}^{n} E_i(f) E_i.$$

Notice that

$$|\nabla f|^2 = \sum_{i=1}^{n} |E_i(f)|^2.$$

We deduce that

$$\Delta f = \sum_{i=1}^{n} g\big(\nabla_{E_i} \sum_{j=1}^{n} E_j(f) E_j, E_i\big) = \sum_{i,j=1}^{n} E_i(E_j(f)) g(E_j, E_i) + E_j(f) g(\nabla_{E_i} E_j, E_i).$$

Evaluating this at p, and remembering that $g(E_i, E_j) = \delta_{ij}$ and $\nabla_{E_i} E_j(p) = 0$, we see that

$$\Delta f(p) = \sum_{i=1}^{n} E_i(E_i(f))(p).$$

Applying this formula with f replaced by f^2 we get, at p,

$$\begin{aligned} \Delta f^2 &= \sum_{i=1}^{n} E_i(E_i(f^2)) = \sum_{i=1}^{n} E_i(2f E_i(f)) \\ &= \sum_{i=1}^{n} 2f(E_i(E_i(f)) + 2(E_i(f))^2 = 2f\Delta f + 2|\nabla f|^2 \end{aligned}$$

as required since p is arbitrary.

(c) We first notice by Cartan's formula that

$$\mathcal{L}_X \Omega = \mathrm{d}(i_X \Omega)$$

since $\mathrm{d}\Omega = 0$.

Let $p \in M$ and take a positively oriented geodesic frame $\{E_1, \ldots, E_n\}$ in $V \ni p$ as in 3(c). Let $\{\xi_1, \ldots, \xi_n\}$ be the dual coframe, i.e. 1-forms on V such that

$$\xi_i(E_j) = \delta_{ij},$$

which means that

$$\xi_i(Y) = g(E_i, Y)$$

for all vector fields Y. Then, clearly,

$$\xi_1 \wedge \ldots \wedge \xi_n = \Omega$$

on V, since both sides are unit length n-forms and we have chosen the geodesic frame to be positively oriented. We then see that

$$i_X\Omega = \sum_{i=1}^{n} (-1)^{(i-1)} \xi_i(X) \xi_1 \wedge \ldots \wedge \widehat{\xi_i} \wedge \ldots \wedge \xi_n$$

where the $\widehat{\xi_i}$ denotes that the term is omitted. Hence,

$$\mathrm{d}(i_X\Omega) = \sum_{i=1}^{n} E_i(\xi_i(X))\xi_1 \wedge \ldots \wedge \xi_n = \sum_{i=1}^{n} E_i(g(E_i, X))\Omega.$$

We finally notice that, at p,

$$\sum_{i=1}^{n} E_i(g(E_i, X))(p) = \sum_{i=1}^{n} g(\nabla_{E_i}E_i, X)(p) + g(E_i, \nabla_{E_i}X)(p) = \operatorname{div} X(p),$$

since $\nabla_{E_i}E_j(p) = 0$ in a geodesic frame, and we use the formula for $\operatorname{div} X$ from our solution to (a). Since p was arbitrary, we obtain the claimed result.

(d) We see from (c) that

$$\int_M (\operatorname{div} X)\Omega = \int_M \mathcal{L}_X\Omega = \int_M \mathrm{d}(i_X\Omega) = 0$$

by Stokes Theorem.

[Note: This is the *Divergence Theorem* in the case when $\partial M = \emptyset$.]

Therefore,

$$\int_M \Delta f \Omega = 0 \quad \text{and} \quad \int_M \Delta(f^2)\Omega = 0$$

as $\Delta = \operatorname{div} \nabla$. Since $\Delta f \geq 0$ on M we deduce from the first equation we must have that

$$\Delta f = 0$$

(i.e. f is *harmonic*). (Note that clearly the same argument would have worked if $\Delta f \leq 0$ on M.) Now using (b) we see that

$$0 = \int_M \Delta(f^2)\Omega = 2\int_M f\Delta f\Omega + 2\int_M |\nabla f|^2\Omega = 2\int_M |\nabla f|^2\Omega.$$

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Since we obviously have $|\nabla f|^2 \geq 0$, we must have $\nabla f = 0$ on M. Hence $\mathrm{d}f = 0$ and thus, since M is connected, f is constant.

[Note: This result can be interpreted as a version of the *maximum principle*, since in this instance it says that a function which is subharmonic ($\Delta f \geq 0$) which has a local interior maximum (this will necessary exist since f is a continuous function on a compact manifold) must in fact be constant.]

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Section C

6. The Euclidean Schwarzschild metric (of mass $m > 0$) is defined for $(\cos\frac{t}{4m}, \sin\frac{t}{4m}) \in \mathcal{S}^1$, $r > 2m$, $\theta \in (0, \pi)$ and $(\cos\phi, \sin\phi) \in \mathcal{S}^1$ by

$$g = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

and extends smoothly to $\theta = 0, \pi$.

(a) Show that there are no geodesics in this metric with r constant.

(b) Show that, given any point p with $r > 2m$ there exists a finite length geodesic γ starting at p ending at a point q with $r = 2m$.

Solution:

(a) We see that any geodesic will satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial r'} - \frac{\partial L}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\left(1 - \frac{2m}{r}\right)^{-1} r'\right)$$
$$- \left(\frac{m}{r^2}(t')^2 - \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-2}(r')^2 + r(\theta')^2 + r\sin^2\theta(\phi')^2\right) = 0.$$

If r is constant then $r' = r'' = 0$ so we have that

$$\frac{m}{r^2}(t')^2 + r(\theta')^2 + r\sin^2\theta(\phi')^2 = 0.$$

All the terms in this expression are non-negative (since $r > 2m > 0$) hence $t' = \theta' = \phi' = 0$, which mean that the geodesic is just a point, which is not a curve.

[Note: The Euclidean Schwarzschild metric is a Euclidean black hole, with horizon at $r = 2m$. This result says that there are no geodesic orbits of the black hole, unlike the usual Schwarzschild metric from General Relativity.]

(b) We give two different methods for solving this question.

Method 1: We can find normalized curves (i.e. curves parametrised by arclength) with t, θ, ϕ all constant by solving the equation

$$\left(1 - \frac{2m}{r}\right)^{-1}(r')^2 = 1$$

and assuming $r' < 0$, i.e.

$$r' = -\left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}.$$

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One may easily check by substituting r' in the geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial r'} - \frac{\partial L}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\left(1-\frac{2m}{r}\right)^{-1} r'\right) - \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-2}(r')^2$$

that we get zero, and hence it does in fact define a geodesic.

Such a geodesic $\gamma : [0, L] \to (M, g)$ will start at a point p with $r = r_0 > 2m$ and end at some point q with the same values for t, θ, ϕ and $r = 2m + \epsilon < r_0$ say.

We want to show that the length of the geodesic is finite as we send $\epsilon \to 0$. We calculate (remembering that $r' < 0$)

$$L(\gamma) = \int_0^L |\gamma'(s)|\mathrm{d}s = \int_L^0 r'\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}}\mathrm{d}s = \int_{2m+\epsilon}^{r_0}\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}}\mathrm{d}r.$$

Since

$$\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}} = \frac{r^{\frac{1}{2}}}{(r-2m)^{\frac{1}{2}}}$$

is integrable on $(2m, r_0)$ we deduce that $L(\gamma)$ remains finite as $\epsilon \to 0$ as claimed.

Method 2: Take the obvious line from p with $r = r_0 > 2m$ to q with $r = 2m$ with the same values of t, θ, ϕ given by $\alpha(s) = (t, r_0 + s(2m - r_0), \theta, \phi)$. Then

$$L(\alpha) = \int_0^1 |\alpha'|\mathrm{d}s = \int_0^1\left(1-\frac{2m}{r_0+s(2m-r_0)}\right)^{-\frac{1}{2}}\mathrm{d}s = \int_0^1 \frac{(r_0-(r_0-2m)s)^{\frac{1}{2}}}{(r_0-2m)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}}\mathrm{d}s$$

which is finite. Reparametrizing α be arc-length gives a curve γ with the same length which now satisfies

$$\left(1-\frac{2m}{r}\right)^{-1}(r')^2 = 1.$$

As before, we have that γ satisfies the geodesic equations and so is a geodesic.

[Note: This result implies that we can reach the horizon of the Euclidean black hole in finite time (as measured along the curve). In fact, our choice of periodicity for t means we can smoothly extend the metric to $r = 2m$ to obtain a metric on a 4-manifold which it is interesting to ask what its topology is: it's a 2-plane bundle over $\mathcal{S}^2$, but one has to check whether it is trivial or not. This metric is also interesting as it is an example of a metric which is Ricci-flat but not flat. The holonomy group of the metric (the group generated by parallel transport) is all of SO(4) – it is a major open question whether there are *any compact* Ricci-flat manifolds with holonomy equal to all of SO(4).]

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