Geometric Group Theory

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A quotation

Leo Moser: “A mathematician named Klein

Thought the Möbius band was divine.

Said he: "If you glue

The edges of two,

You’ll get a weird bottle like mine.”

Indeed, by glueing two Möbius bands one obtains a Klein bottle.

There is a second way of obtaining a Klein bottle, that relates to a construction we will introduce this week: the HNN extension.

Theorem

$G = A *_H B$ acts on a tree T with fundamental domain an edge $[P, Q]$ such that $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.

Proof:

Let $V(T) = G/A \sqcup G/B$.

Edges are (gA, gB), i.e. we join two left cosets of A and B if they have a common representative g. Given an edge, what is the set of common representatives corresponding to it?

$$g_1 A = gA, \quad g_1 B = gB \iff g^{-1} g_1 \in A \cap B = H$$

So the set is exactly gH. We label the edge (gA, gB) by gH and the edge (gB, gA) by $g\bar{H}$. Clearly, G acts transitively on the (non-oriented) edges and there are two orbits of vertices.

Let T be the non-oriented graph.

T is connected: For each edge $\{gA, gB\}$, $g = hs_1...s_n$, we will prove it is connected by an edge path to $\{A, B\}$, by induction on n. Moreover, the length of the edge path (including $\{A, B\}$ and $\{gA, gB\}$) is $n+1$. The case $n = 0$ is obvious.

Induction: if $s_n \in A_1 \setminus \{1\}$ then

$$gA = \underbrace{hs_1...s_{n-1}}_{g'} A$$

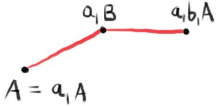
and $\{gA, gB\}$ shares an endpoint with $\{g'A, g'B\}$.
Similarly, if $s_n \in B_1 \setminus \{1\}$ then $gB = hs_1...s_{n-1}B$ and $\{gA, gB\}$ has a common endpoint with $\{g'A, g'B\}$.

Amalgams and actions on trees

T is a tree: A path without spikes in T of origin A and even length $2n$ has vertices of the form:

$$A = a_1 A, a_1 B, a_1 b_1 A, ..., a_1 b_1 ... a_n b_n A$$

where $a_i \notin H$ and $b_i \notin H$.



An easy induction on n shows that the reduced form of $a_1 b_1 ... a_n b_n$ is $h a_1' b_1' ... a_n' b_n'$: for $n = 1$ we have

$$a_1 b_1 = a_1 \underbrace{h b_1'}_{b_1' \neq 1 \text{ as } b_1 \notin H} = h' a_1' b_1' \quad \text{where} \quad a_1', b_1' \neq 1$$

Amalgams and actions on trees

Likewise,

$$a_1 b_1 a_2 b_2 \ldots a_{n+1} b_{n+1} = a_1 b_1 h a_2' b_2' \ldots a_{n+1}' b_{n+1}' = h' a_1' b_1' \ldots a_{n+1}' b_{n+1}'$$

In particular we cannot have $a_1 b_1 \ldots a_n b_n A = A$ otherwise

$$\underbrace{h a_1' b_1' \ldots a_n' b_n'}_{\text{length } 2n} = \underbrace{h' a'}_{\text{length } 0 \text{ or } 1}$$

So there is no cycle through A and so there is no cycle in T (every cycle must contain one vertex in G/A and so can be G-translated to a cycle through A). □

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Corollary

If $F \leq A *_H B$ is such that $F \cap gAg^{-1} = \{1\}$ and $F \cap gBg^{-1} = \{1\}$ for every $g \in G$, then F is free.

Proof.

F acts on the tree T without vertex or edge stabilisers and so it acts freely on the tree T. □

Proposition

The kernel of the map $\varphi : A * B \to A \times B$ is free.

Amalgams and actions on trees

Corollary

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Proposition

The kernel of the map $\varphi : A * B \to A \times B$ is free.

Proof.

$F = \ker \varphi$ intersects gAg^{-1} and gBg^{-1} trivially. □

Corollary

If A, B finite, then $A * B$ is virtually free.

A group is virtually (*) if it has a finite index subgroup that is (*).

Amalgams and actions on trees

Theorem

Suppose $G \curvearrowright T$ with fundamental domain an edge $e = [P, Q]$. If $A = \mathrm{Stab}(P)$, $B = \mathrm{Stab}(Q)$, $H = \mathrm{Stab}(e)$ then $G = A *_H B$.

Proof

Since we have $\alpha_1 : A \to G$, $\beta_1 : B \to G$ agreeing on H, there exists some $\varphi : A *_H B \to G$, by the Universal Property of $A *_H B$.

Step 1: $G = \langle A, B \rangle$, that is, φ is onto.

For all $g \in G$, ge is joined to e by a unique path of length n (counting e and ge). We will prove that $g \in \langle A, B \rangle$ by induction on n. If $n = 1$ then $g \in H$.

Assume true for n, and let ge be joined to e by a path of length $n + 1$. Let $g'e$ be the previous edge on the path.

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Then either $gP = g'P$ or $gQ = g'Q$ and so $g^{-1}g' \in A \cup B$. Since $g' \in \langle A, B\rangle$ we are done.

Step 2: φ is injective.

Let $hs_1 \ldots s_n \in \ker \varphi$. We can prove by induction on n that $hs_1 \ldots s_n e$ can be joined to e by an edge path with no spikes of length $n+1$. Hence $hs_1 \ldots s_n \neq 1$ in G. □