C3.11 Riemannian Geometry

Sheet 1 — HT24

Solutions

This problem sheet is based on Sections 1 and 2 of the lecture notes. This version contains solutions to Sections A and C.

Section A

1. Let
$$B^2 = \{(y_1, y_2) \in \mathbb{R}^2 \,:\, y_1^2 + y_2^2 < 1\}$$
and define a Riemannian metric g on B^2 by
$$g = \frac{4\mathrm{d}y_1^2 + 4\mathrm{d}y_2^2}{(1 - y_1^2 - y_2^2)^2}.$$
Let
$$H^2 = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 > 0\}$$
and define a Riemannian metric h on H^2 by
$$h = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{x_2^2}.$$
   (a) Define $f : B^2 \to H^2$ by
   $$f(y_1, y_2) = \frac{(2y_1, 1 - y_1^2 - y_2^2)}{y_1^2 + (y_2 + 1)^2}$$
   Show that f is a diffeomorphism.
   [*Hint: What is $f \circ f$?*]
   (b) Compute $f_*(\partial_1)$ and $f_*(\partial_2)$.
   (c) Hence (or otherwise) deduce that $f^*h = g$.

Solution:

(a) Obviously f is smooth. One can see that if one writes $f(y_1, y_2) = (x_1, x_2)$ then
$$\begin{aligned}
x_1^2 + (x_2 + 1)^2 &= \frac{4y_1^2 + (1 - y_1^2 - y_2^2 + y_1^2 + (y_2 + 1)^2)^2}{(y_1^2 + (y_2 + 1)^2)^2} \\
&= \frac{4y_1^2 + 4(y_2 + 1)^2}{(y_1^2 + (y_2 + 1)^2)^2} \\
&= \frac{4}{y_1^2 + (y_2 + 1)^2}.
\end{aligned}$$

Therefore,

$$\frac{2x_1}{x_1^2+(x_2+1)^2} = \frac{4y_1}{y_1^2+(y_2+1)^2}\frac{y_1^2+(y_2+1)^2}{4} = y_1.$$

Similarly, one finds that

$$\begin{aligned} 1-x_1^2-x_2^2 &= \frac{(y_1^2+(y_2+1)^2)^2-4y_1^2-(1-y_1^2-y_2^2)^2}{(y_1^2+(y_2+1)^2)^2} \\ &= \frac{4y_2(y_1^2+(y_2+1)^2)}{(y_1^2+(y_2+1)^2)^2} \\ &= \frac{4y_2}{y_1^2+(y_2+1)^2}, \end{aligned}$$

so

$$\frac{1-x_1^2-x_2^2}{x_1^2+(x_2+1)^2} = y_2.$$

Hence,

$$f(f(y_1,y_2)) = (y_1,y_2)$$

so f is invertible with $f^{-1}=f$.

As f is smooth and invertible and its own inverse, f is a diffeomorphism.

[Note: If we let $z = y_1 - iy_2$ and $w = x_1 + ix_2$ then the map f is equivalent to the Möbius transformation

$$z \mapsto i\left(\frac{1-z}{z-i}\right).]$$

(b) We compute the differential of f as:

$$\mathrm{d}f_y = \frac{2}{(y_1^2+(y_2+1)^2)^2}\begin{pmatrix} (y_2+1)^2-y_1^2 & -2y_1(y_2+1) \\ -2y_1(y_2+1) & y_1^2-(y_2+1)^2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} f_*(\partial_1) &= \frac{2\big((y_2+1)^2-y_1^2\big)\partial_1 - 4y_1(y_2+1)\partial_2}{(y_1^2+(y_2+1)^2)^2} \\ f_*(\partial_2) &= \frac{-4y_1(y_2+1)\partial_1 + 2\big(y_1^2-(y_2+1)^2\big)\partial_2}{(y_1^2+(y_2+1)^2)^2} \end{aligned}$$

(c) If we continue to write $f(y_1,y_2) = (x_1,x_2)$ then

$$\begin{aligned} f^*h(\partial_1,\partial_1) &= h(f_*\partial_1, f_*\partial_1) \\ &= \frac{1}{x_2^2}\frac{4\big((y_2+1)^2-y_1^2\big)^2+16y_1^2(y_2+1)^2}{(y_1^2+(y_2+1)^2)^4} \\ &= \frac{(y_1^2+(y_2+1)^2)^2}{(1-y_1^2-y_2^2)^2}\frac{4(y_1^2+(y_2+1)^2)^2}{(y_1^2+(y_2+1)^2)^4} \\ &= \frac{4}{(1-y_1^2-y_2^2)^2} \\ &= g(\partial_1,\partial_1) \end{aligned}$$

By the formulae for $f_*(\partial_2)$, we deduce that

$$f^*h(\partial_2, \partial_2) = \frac{4}{(1 - y_1^2 - y_2^2)^2} = g(\partial_2, \partial_2)$$

as well. Finally,

$$\begin{aligned} f^*h(\partial_1, \partial_2) &= h(f_*\partial_1, f_*\partial_2) \\ &= \frac{1}{x_2^2} \frac{-8\big((y_2+1)^2 - y_1^2\big)y_1(y_2+1) - 8y_1(y_2+1)\big(y_1^2 - (y_2+1)^2\big)}{(y_1^2 + (y_2+1)^2)^4} \\ &= 0 = g(\partial_1, \partial_2). \end{aligned}$$

Overall, $f^*h = g$, so f is an isometry.

2. Let (M, g) be a connected Riemannian manifold and let $\widetilde{M}$ be the universal cover of M.
   (a) Show that there exists a unique Riemannian metric $\tilde{g}$ on $\widetilde{M}$ such that the covering map $\pi : (\widetilde{M}, \tilde{g}) \to (M, g)$ is a local isometry.
   (b) Show that the fundamental group of M acts on $(\widetilde{M}, \tilde{g})$ by isometries.

Solution:

(a) Since $\pi : \widetilde{M} \to M$ is a local diffeomorphism, it is (in particular) an immersion and so $\pi^*g = \tilde{g}$ is a Riemannian metric on $\widetilde{M}$. By construction, $\pi : (\widetilde{M}, \tilde{g}) \to (M, g)$ is a local isometry.

If h were any other Riemannian metric on $\widetilde{M}$ so that $\pi : (\widetilde{M}, h) \to (M, g)$ were a local isometry, then for every point $\tilde{p} \in \widetilde{M}$ there would exist open sets $\widetilde{U} \ni \tilde{p}$ and $U \ni \pi(\tilde{p})$ so that $\pi^*g = h$ on $\widetilde{U}$, and so $h = \tilde{g}$ at $\tilde{p}$. Hence, $\tilde{g}$ is unique.

(b) By definition of the universal cover there is an action of the fundamental group $\pi_1(M)$ on $\widetilde{M}$ given by $a \in \pi_1(M) \mapsto f_a \in \mathrm{Diff}(\widetilde{M})$ so that $\pi \circ f_a = \pi$ for all $a \in \pi_1(M)$. Hence, $\pi^* = f_a^* \circ \pi^*$ and so

$$\tilde{g} = \pi^*g = f_a^* \circ \pi^*g = f_a^*\tilde{g}.$$

We deduce that $f_a \in \mathrm{Isom}(\widetilde{M}, \tilde{g})$ for all $a \in \pi_1(M)$ as desired.

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Section B

3. Let

$$\mathcal{H}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n} x_j^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}$$

and let g be the restriction of

$$h = \sum_{j=1}^{n} \mathrm{d}x_j^2 - \mathrm{d}x_{n+1}^2$$

on $\mathbb{R}^{n+1}$ to $\mathcal{H}^n$.

(a) Show that g is a Riemannian metric on $\mathcal{H}^n$.

(b) Let $f(x) = Ax$ be a linear map on $\mathbb{R}^{n+1}$ given by $A = (a_{ij}) \in M_{n+1}(\mathbb{R})$ and let

$$G = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. Show that f defines an isometry on $(\mathcal{H}^n, g)$ if and only if

$$A^{\mathrm{T}} G A = G \quad \text{and} \quad a_{n+1,n+1} > 0.$$

(c) Now let $n = 2$, $L > 0$ and $\alpha : [0, L] \to \mathcal{H}^2$ be given by $\alpha(t) = (\sinh t, 0, \cosh t)$. If $\tau_\alpha : T_\alpha(0)\mathcal{H}^2 \to T_{\alpha(L)}\mathcal{H}^2$ is the parallel transport map, compute $\tau_\alpha(\partial_1)$ and $\tau_\alpha(\partial_2)$.

4. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds with Levi-Civita connections ∇_1 and ∇_2 respectively. Recall that $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \times T_{p_2}M_2$ for all $(p_1, p_2) \in M_1 \times M_2$. Define g on $M_1 \times M_2$ by

$$g_{(p_1,p_2)}((X_1, X_2), (Y_1, Y_2)) = (g_1)_{p_1}(X_1, Y_1) + (g_2)_{p_2}(X_2, Y_2).$$

(a) Show that g is a Riemannian metric on $M_1 \times M_2$.

(b) Show that the Levi-Civita connection ∇ of g on $M_1 \times M_2$ satisfies

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) = \big((\nabla_1)_{X_1}Y_1, (\nabla_2)_{X_2}Y_2\big)$$

for all vector fields X_1, Y_1 on M_1 and X_2, Y_2 on M_2.

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5. Let (H^2, h) be the upper half-space with the hyperbolic metric

$$h = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{x_2^2}.$$

(a) Calculate the Christoffel symbols of h in the coordinates (x_1, x_2) on H^2 using the definition or formula for the Christoffel symbols.

Let $\alpha : [0, L] \to (H^2, h)$ be the curve $\alpha(t) = (t, 1)$ and let τ_α be the parallel transport along α.

(b) Let $X_0 = \partial_2 \in T_{(0,1)}H^2$. Calculate $\tau_\alpha(X_0)$ and show that, viewed as a vector in Euclidean $\mathbb{R}^2$, it makes an angle L with the vertical axis.

Let

$$G = \{u : \mathbb{R} \to \mathbb{R} : u(x_1, x_2)(t) = x_1 + tx_2, x_1 \in \mathbb{R},\ x_2 > 0\}$$

and define a manifold structure on G so that $f : G \to H^2$ given by $f(u(x_1, x_2)) = (x_1, x_2)$ is a diffeomorphism. Define a Riemannian metric g on G by $g = f^*h$.

(c) Show that, for all $v \in G$, the map $L_v : G \to G$ given by $L_v(u) = v \circ u$ is an isometry of g.

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Section C

6. Let $\mathcal{S}^2$ be the unit sphere in $\mathbb{R}^3$ endowed with the round metric g, let $U = \mathcal{S}^2 \setminus \{(0,0,1)\}$ and let $\varphi : U \to \mathbb{R}^2$ be

$$\varphi(x_1, x_2, x_3) = \frac{(x_1, x_2)}{1 - x_3}$$

so that

$$\varphi^{-1}(y_1, y_2) = \frac{(2y_1, 2y_2, y_1^2 + y_2^2 - 1)}{y_1^2 + y_2^2 + 1}$$

(a) Show that

$$(\varphi^{-1})^* g = \frac{4(\mathrm{d}y_1^2 + \mathrm{d}y_2^2)}{(1 + y_1^2 + y_2^2)^2}.$$

Let $\beta : [0, 2\pi] \to \mathbb{R}^2$ be given by $\beta(t) = (\cos t, \sin t)$.

(b) Using the fact that $\varphi^{-1} : (\mathcal{S}^2 \setminus \{0,0,1\}, g) \to (\mathbb{R}^2, (\varphi^{-1})^* g)$ is an isometry or otherwise, show that the restrictions of the vector fields

$$y_1 \partial_1 + y_2 \partial_2 \quad \text{and} \quad -y_2 \partial_1 + y_1 \partial_2$$

to β are parallel along β with respect to the metric $(\varphi^{-1})^* g$.

Solution:

(a) We compute directly that

$$\begin{aligned}(\varphi^{-1})^* g &= \mathrm{d}(x_1 \circ \varphi^{-1})^2 + \mathrm{d}(x_2 \circ \varphi^{-1})^2 + \mathrm{d}(x_3 \circ \varphi^{-1})^2 \\ &= \mathrm{d}\left(\frac{2y_1}{y_1^2 + y_2^2 + 1}\right)^2 + \mathrm{d}\left(\frac{2y_2}{y_1^2 + y_2^2 + 1}\right)^2 + \mathrm{d}\left(\frac{y_1^2 + y_2^2 - 1}{y_1^2 + y_2^2 + 1}\right)^2.\end{aligned}$$

We see that

$$\begin{aligned}\mathrm{d}\left(\frac{2y_1}{y_1^2 + y_2^2 + 1}\right) &= \frac{2(y_1^2 + y_2^2 + 1)\mathrm{d}y_1 - 2y_1 \mathrm{d}(y_1^2 + y_2^2 + 1)}{(y_1^2 + y_2^2 + 1)^2} \\ &= \frac{2(y_2^2 - y_1^2 + 1)\mathrm{d}y_1 - 4y_1 y_2 \mathrm{d}y_2}{(y_1^2 + y_2^2 + 1)^2} \\ \mathrm{d}\left(\frac{2y_2}{y_1^2 + y_2^2 + 1}\right) &= \frac{2(y_1^2 + y_2^2 + 1)\mathrm{d}y_2 - 2y_2 \mathrm{d}(y_1^2 + y_2^2 + 1)}{(y_1^2 + y_2^2 + 1)^2} \\ &= \frac{-4y_1 y_2 \mathrm{d}y_1 + 2(y_1^2 - y_2^2 + 1)\mathrm{d}y_2}{(y_1^2 + y_2^2 + 1)^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{d}\left(\frac{2y_1}{y_1^2 + y_2^2 + 1}\right)^2 + \mathrm{d}\left(\frac{2y_2}{y_1^2 + y_2^2 + 1}\right)^2 \\ &\qquad = \frac{4(y_2^2 - y_1^2 + 1)^2 \mathrm{d}y_1^2 - 32 y_1 y_2 \mathrm{d}y_1 \mathrm{d}y_2 + 4(y_1^2 - y_2^2 + 1)^2 \mathrm{d}y_2^2}{(y_1^2 + y_2^2 + 1)^4}.\end{aligned}$$

We then compute

$$\begin{aligned}
\mathrm{d}\left(\frac{y_1^2+y_2^2-1}{y_1^2+y_2^2+1}\right) &= \mathrm{d}\left(1-\frac{2}{y_1^2+y_2^2+1}\right) \\
&= \frac{2\mathrm{d}(y_1^2+y_2^2+1)}{(y_1^2+y_2^2+1)^2} \\
&= \frac{4y_1\mathrm{d}y_1+4y_2\mathrm{d}y_2}{(y_1^2+y_2^2+1)^2}
\end{aligned}$$

and hence

$$\mathrm{d}\left(\frac{y_1^2+y_2^2-1}{y_1^2+y_2^2+1}\right)^2 = \frac{16(y_1^2\mathrm{d}y_1^2+2y_1y_2\mathrm{d}y_1\mathrm{d}y_2+y_2^2\mathrm{d}y_2^2)}{(y_1^2+y_2^2+1)^4}$$

Adding our results cancels the $\mathrm{d}y_1\mathrm{d}y_2$ term and gives

$$\begin{aligned}
(\varphi^{-1})^*g &= \frac{4(y_1^2+y_2^{+}1)^2(\mathrm{d}y_1^2+\mathrm{d}y_2^2)}{(y_1^2+y_2^2+1)^4} \\
&= \frac{4(\mathrm{d}y_1^2+\mathrm{d}y_2^2)}{(y_1^2+y_2^2+1)^2}
\end{aligned}$$

as required.

(b) If we choose coordinates

$$(x_1,x_2,x_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = f(\theta,\phi)$$

as usual, we see that

$$\gamma(t) = \varphi^{-1}\circ\beta(t) = (\cos t, \sin t, 0) = f(\frac{\pi}{2}, t),$$

which is the equator in $\mathcal{S}^2$. Hence if we let $X_1 = f_*(\partial_\theta)$ and $X_2 = f_*(\partial_\phi)$ then along β we have

$$Y_2 = (\varphi^{-1})_*(X_2) = \beta' = -\sin t\partial_1 + \cos t\partial_2 = -y_2\partial_1 + y_1\partial_2.$$

Hence Y_2 is parallel along β as X_2 is parallel along $\gamma = \varphi^{-1}\circ\beta$ and φ^{-1} is an isometry by definition.

We also know that X_1 is parallel along γ and X_1 is unit length and orthogonal to X_2 along γ, so any unit vector field along β which is orthogonal to Y_2 must be parallel (since it must be the pushforward of X_1 up to sign). However, we see that

$$Y_1 = \cos t\partial_1 + \sin t\partial_2 = x_1\partial_1 + x_2\partial_2$$

along β is unit length and orthogonal to Y_2 since along β

$$(\varphi^{-1})^*g|_\beta = \mathrm{d}y_1^2 + \mathrm{d}y_2^2$$

as $y_1^2+y_2^2 = 1$ along β. Thus Y_1 is also parallel along β.

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