Chapter 6
Fiber products & all that jazz

First, we will introduce few related notions.

def. 1) $f: X \to Y \in Sch$ is an open immersion if f induces an isom onto an open subscheme of Y, i.e. $(U, \mathcal{O}_{Y}|_{U})$, $U \subseteq Y$ open.

2) $g: X \to Y$ is a closed immersion if g is a homeo onto a closed subset of Y and $g^{\#}: \mathcal{O}_Y \to g_* \mathcal{O}_X$ is surjective.

Ex: $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[t]/t^n \hookrightarrow \operatorname{Spec} k[t]$

3) a closed subscheme of Y is an equivalence class of closed immersions to Y, where $[X \to Y] \sim [X' \to Y]$ iff there's a triangle

$$\begin{tikzcd} X' \arrow[rr, "\cong"] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

(we will see a more explicit def. later!)

This definition is in the same spirit as the following one:

def. $S \in Sch$. An S-scheme is a scheme X with a chosen map $X \to S$, called structure morphism.

A morphism of S-schemes is a comm. diagr.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

$\rightsquigarrow$ define the category Sch_S.

Abbreviate: $Sch_{\operatorname{Spec} A} =: Sch_A$.

$\Downarrow$

scheme X with A-algebra structure on $\mathcal{O}_X(X)$

Ex: $Sch = Sch_{\mathbb{Z}}$.

§ Fiber products

Motivation: fiber products help us to

- define the right notion of product in the category of S-schemes
- $X_1 \hookrightarrow Y$, $X_2 \hookrightarrow Y$ closed subschemes $\rightsquigarrow$ define „$X_1 \cap X_2$" as a scheme
- $f: X \to Y$, $y \in Y \rightsquigarrow$ define „$f^{-1}(y)$" as a scheme
- obtain $\mathbb{P}^n_R$ from $\mathbb{P}^n_{\mathbb{Z}}$ and $\mathbb{Z} \hookrightarrow R$. (e.g., $R = \mathbb{C}$).

def. Let $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ be morphisms.
The **fiber product** is a scheme $X \times_S Y$ with maps

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$ commuting

notation for a square being a fiber product

which is universal:

for any commuting

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & S \end{tikzcd}$$

s.t. the diagram commutes.

Rem. 1) If $X \times_S Y$ exists, it's unique up to unique isom.

2) Fiber products make sense in any category (but may not exist).

In sets: $X \times_S Y \subseteq X \times Y$ is the subset of pairs (x,y) s.t. $f(x) = g(y) \in S$.

Thm. Fiber products of scheme exists, i.e. $\forall X, Y \in Sch_S$ $\exists (!)$ $X \times_S Y \in Sch_S$ with this universal property.

Rem: often $S = \operatorname{Spec} \mathbb{Z} \rightsquigarrow X \times Y \in Sch$

or $S = \operatorname{Spec} k \rightsquigarrow X \times Y \in Sch_k$

($k = \bar{k}$, X, Y k-varieties $\Rightarrow X \times_k Y$ also a k-variety)

As a set, $X \times Y$ is the Cartesian product, but it has a different topology!

NB: $\mathbb{A}^n \simeq \mathbb{A}^1 \times \ldots \times \mathbb{A}^1$ where $\times$ is the product in Sch (fiber product), not Cartesian product:

e.g. $\{x \cdot y = 1\} \subset \mathbb{A}^2$ is a closed subset, but it wouldn't be closed in Cartesian product top.

Proof sketch of thm [Hartshorne Thm 3.3]

① affine case

if X, Y, S are affines associated to rings A, B, R then $\operatorname{Spec} A \otimes_R B$ does the job. ✓

Because: $Z \to \operatorname{Spec} A \otimes_R B \leftrightsquigarrow A \otimes_R B \to \Gamma(Z, \mathcal{O}_Z)$ ring map,

which is uniquely determined by $\begin{matrix} A \searrow \\ B \nearrow \end{matrix} \Gamma(Z, \mathcal{O}_Z)$ as R-maps.

Globalization: slowly turn X, Y, S into affines

② if $X \times_S Y$ exists and $U \subseteq X$ open, then $U \times_S Y$ exists: take $p_X^{-1}(U)$ with open subscheme structure.

③. if $X = \bigcup U_i$ and $U_i \times_S Y$ exist, then they can be glued together to $X \times_S Y$ (glue U_i's to X and glue maps to Y, cocycle condition is automatic)

④ 1, 2, 3 $\Rightarrow$ when Y, S affine $X \times_S Y$ exists $\forall S$.
symmetric in X & Y $\Rightarrow X \times_S Y$ exists $\forall$ affine.

⑤ let $S = \bigcup S_i$ affine open cover.
Let $X_i := f^{-1}(S_i)$, $Y_i := g^{-1}(S_i)$. $X_i \times_{S_i} Y_i$ exist.

Note: $X_i \times_{S_i} Y_i = X_i \times_S Y$ (obvious for sets)

⑥ glue again and you win!

§ Examples of fiber products

1) **base change** (generalizes the idea of changing coeffs of equations)

$\mathbb{A}^n_R = \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R$; „$\mathbb{A}^n_{\mathbb{C}}$ is base change of $\mathbb{A}^n_{\mathbb{Z}}$ to $\mathbb{C}$"

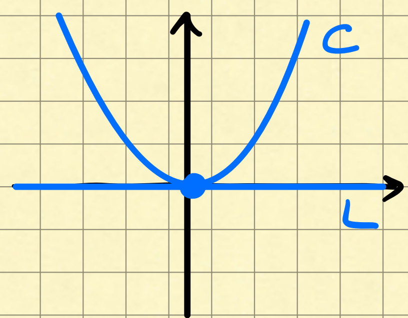
$\mathbb{P}^n_R := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R$. Also works $\forall\ \mathbb{A}^n_X, \mathbb{P}^n_X$ (X non-affine!)

Actually, $\forall T, X$ S-schemes $T_X := T \times_S X$ – base change of T to X

2) **intersections**

$C := \operatorname{Spec} \mathbb{C}[x,y]/(y-x^2)$

$L := \operatorname{Spec} \mathbb{C}[x,y]/(y)$



$C \times_{\mathbb{A}^2} L = \operatorname{Spec} \mathbb{C}[x]/x^2$ – „double point"!

this is the correct notion of intersection :)

3) **deformations**

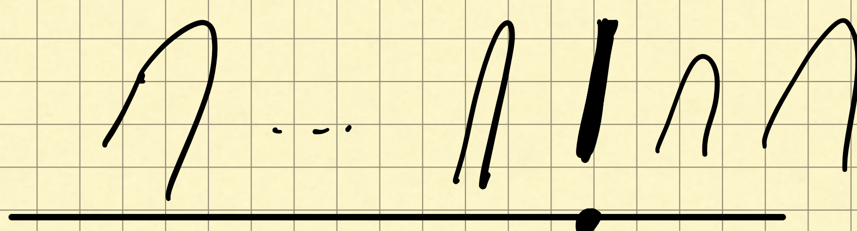
$$\begin{array}{ccc} \operatorname{Spec} \mathbb{C}[x,y]/y^2 & \longrightarrow & \operatorname{Spec} \mathbb{C}[x,y,t]/y^2+tx \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{C}[t]/(t) & \longrightarrow & \operatorname{Spec} \mathbb{C}[t] \end{array}$$

„double x-axis"

family of mostly conics

closed pt 0

generic picture of a deformation family:



More generally:

4) (schematic) fibers

recall that $\forall p \in S$ we have

$$\operatorname{Spec} \underset{\substack{\parallel \\ A_p/p\cdot A_p}}{\kappa(p)} \hookrightarrow \underset{\text{affine open}}{\operatorname{Spec} A} \subset S$$

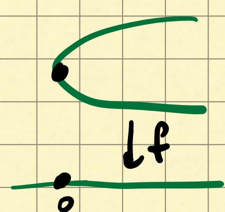
def. For any $X \xrightarrow{\varphi} S$ the (scheme-theoretic) fiber of φ at $p \in S$ is

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ \operatorname{Spec} \kappa(p) \arrow[r] & S \end{tikzcd}$$

Ex: $k = \bar{k}$ $\quad f : \mathbb{A}^1_k \to \mathbb{A}^1_k$

$k[y] \leftarrow k[x]$

$y^2 \leftarrow\!\shortmid x$



fiber over 0: $\operatorname{Spec} k \underset{k[x]}{\otimes} k[y] =$

$= \operatorname{Spec} k[y]/(y^2)$ „double point"!

X_p is a scheme over $\kappa(p)$!

NB: different fibers live over different fields

5) generic fiber

aka fiber over generic point:

encodes „general" behaviour (which happens over a dense open)

Ex: in 3) the generic fiber is

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C}(t)[x,y]/(y^2+tx) \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{C}[x,y,t]/(y^2+tx) \arrow[d] \\ \operatorname{Spec} \mathbb{C}(t) \arrow[r] & \operatorname{Spec} \mathbb{C}[t] = S \end{tikzcd}$$

$t \neq 0$ so it's a family of conics

§ Separatedness

X scheme $\rightsquigarrow X_{top} \in \mathrm{Top}$ is not Hausdorff usually (generic points!)

But: we want an analogue of Hausdorfness that makes sense for schemes

In topology: X Hausdorff $\iff \Delta_X \subset X \times X$ is closed

can separate any pair of pts using open nbhds

product top.

In geometry: define the diagonal map to be

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta_{X/S}"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X \times_S X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

when S not specified means $S = \mathrm{Spec}\,\mathbb{Z}$

def. A map $X \to S$ (or $X \in \mathrm{Sch}_S$) is separated if $\Delta_{X/S}$ is a closed immersion.

Fact: enough to check that $\mathrm{im}(X) \subset X \times_S X$ is a closed subset.

Ex: 1) $X \to S$ map of affine schemes $\Rightarrow$ it is separated, because $A \otimes_B A \to A$ is always surjective exercises

2) $\mathbb{A}^n_S$, $\mathbb{P}^n_S$ are separated S-schemes $\forall$ affine S

3) open & closed embeddings are separated

4) compositions of separated maps are separated

so, almost any scheme is separated except particularly bad ones:

5) the bug-eyed line $\mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ is NOT separated! (exercise)

People typically work with separated schemes, to avoid pathologies like the bug-eyed line.

§ Varieties

def. A k-scheme X is of finite type if $X = \bigcup_{j=1}^{m} \operatorname{Spec} A_j$ for some fin. gen. k-algebras A_j.

In other words, X is quasicompact and $\mathcal{O}_X(U)$ are f.g. k-algebras.

def. $k = \bar{k}$. A variety over k is a reduced, finite type, separated k-scheme (some sources also require „irreducible")

equivalent to varieties defined via regular functions!

Rem. all quasi-projective varieties from classical alg geom are varieties, but $\exists$ variety $\not\hookrightarrow \mathbb{P}^n_k$ (Nagata).