Chapter 7
Properness & Flatness

We have already seen a bunch of types of morphisms:

open immersion, closed immersion, separated, ...

We will now learn few more properties of morphisms & some techniques to prove them.

§ Properness

In topology: X compact $\Leftrightarrow$ $\forall Y$ $X \times Y \xrightarrow{pr} Y$ sends closed sets to closed sets

def. 1) A map of schemes $f: X \to S$ is **closed** if $\forall Z \subseteq X$ closed subset $f(Z) \subseteq S$ is closed.

2) A map is **universally closed** if $\forall$ base change of it is closed too:

$$\begin{tikzcd} X \times_S T \arrow[r] \arrow[d, "\text{also closed}"'] & X \arrow[d, "f"] \\ T \arrow[r, "\forall g"'] & S \end{tikzcd}$$

„relative compactness"

Non-example: $\mathbb{A}^1_k \to \operatorname{Spec} k$ is closed but its base change $\mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$, $(a, b) \mapsto b$ is not: $\underset{\text{closed}}{Z(xy-1)} \mapsto \underset{\text{not closed}}{\mathbb{A}^1_k - 0}$.

Rem. Many properties of morphisms are preserved under base change (e.g. separated), and we tend to prefer such properties.

def. A map of schemes $f: X \to Y$ is **proper** if it is universally closed, separated and finite type. A scheme X is proper if $X \to \operatorname{Spec} \mathbb{Z}$ is proper.

We have seen finite type k-schemes, in general it means „gluing from spectra of f.g. algebras":

def. $f: X \to Y$ is **of finite type** if $\exists$ open cover $Y = \bigcup V_i$, $V_i = \operatorname{Spec} B_i$ s.t. $\forall i$ $f^{-1}(V_i)$ has a finite open cover by $\operatorname{Spec} A_{ij}$ where each A_{ij} is a f.g. B_i-algebra.

Rem. Properness is stable under base change: $f: X \to Y$ proper $\Rightarrow \forall\, Z \to Y$ $\quad Z \times_Y X \to Z$ proper

Ex: $\mathbb{A}^n_R \to \operatorname{Spec} R$ not proper;
$\mathbb{P}^n_R \to \operatorname{Spec} R$ proper $\quad$ „compactification of $\mathbb{A}^n_R$"
(more on that — Below!)

These 3 conditions together seem tedious to check. But luckily, there's a cool criterion which works when you add an extra mild condition on X — noetherianity.

§ Valuative criterion

def. A scheme X is **Noetherian** if
$X = \bigcup_{i=1}^{m} \operatorname{Spec} A_i$, A_i Noetherian rings
(all ideals of A_i are fin. gen.)

Thm. (Valuative criterion of properness)
Let $f: X \to Y$ map of schemes, X Noetherian.
Then f is proper iff $\forall$ dvr A with $\operatorname{Frac}(A) = K$
given

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

$\approx$ arithmetic criterion for geom./top. property!

We won't prove it but
we will see how to use it in examples.

Reminder on dvr's:

- examples: $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$, $k[[T]]$, local rings of points on smooth curves
- it's a PID with one non-zero max ideal m
- it has a uniformizer π: $m = (\pi)$
and any ideal is (π^k), $k \in \mathbb{N}$
- $\forall a \in A$ can be written as $a = u \cdot \pi^k$, $u \in A^\times = A - m$
- $\forall t \in K$ is $t = u \cdot \pi^k$, $u \in A^\times$, $k \in \mathbb{Z}$

Applications.

① $\mathbb{P}^n_{\mathbb{Z}}$ is proper (hence $\mathbb{P}^n_R \to \operatorname{Spec} R$ $\forall R$ is proper by base change)

pick A a dvr, $\operatorname{Frac}(A) = K$, π uniformizer
want: $\mathbb{P}^n_{\mathbb{Z}}(K) \leftrightarrow \mathbb{P}^n_{\mathbb{Z}}(A)$ bijection

$\mathbb{P}^n_{\mathbb{Z}}(A) = \operatorname{Hom}_{Sch}(\operatorname{Spec} A, \mathbb{P}^n_{\mathbb{Z}})$ „A-points of $\mathbb{P}^n_{\mathbb{Z}}$"

A K-point of $\mathbb{P}^n_{\mathbb{Z}}$ is $\underbrace{[z_0 : \ldots : z_n]}_{\text{up to a scalar}}$ $z_i \in K$, not all zero

Let $z_i' := \underbrace{\pi^m}_{\in K^\times} \cdot z_i$, pick $m \in \mathbb{N}$ so that $z_i' \in A$,
then $[z_0 : \ldots : z_n] = [z_0' : \ldots : z_n'] \leftarrow$ A-point

② $\mathbb{A}^n_k \to \operatorname{Spec} k$ is not proper $\forall n$:
take $A = k[[T]]$, $K = k((T))$
consider $\operatorname{Spec} K \to \mathbb{A}^n_k$ given by $(\frac{1}{T}, 1, 1, \ldots, 1)$.
It cannot be extended to an A-point!
Because $1/T \notin A$.

③ $X \hookrightarrow \mathbb{P}^n$ closed $\Rightarrow$ $X \to \operatorname{Spec} \mathbb{Z}$ proper:

$$\begin{array}{ccc} X(A) & \hookrightarrow & \mathbb{P}^n(A) \\ \text{bijective} \updownarrow & \Leftarrow & \updownarrow - \text{bijective} \\ X(K) & \hookrightarrow & \mathbb{P}^n(K) \end{array}$$

<u>def.</u> 1) A morphism $f: X \to Y$ is **projective** if it can be factored as

$$X \underset{\text{closed immersion}}{\hookrightarrow} \mathbb{P}^m_Y \xrightarrow[\text{pr}]{} Y$$

Fact: Y Noetherian $\Rightarrow$ such f is proper, and most proper map arise this way.

2) $f: X \to Y$ is **quasi-projective** if it can be factored as

$$X \underset{\text{open imm}}{\hookrightarrow} Z \xrightarrow[\text{projective}]{} Y$$

Fact. X, Y Noetherian $\Rightarrow$ it's equivalent to being finite type & separated (most maps are quasi-projective)

§ Flatness

Spoiler: properness & flatness are important properties for later chapters (sheaves of modules ...)

morally: flat maps encode continuously varying families

def. A map of schemes $f: X \to Y$ is flat if all $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are flat ring homs, i.e. $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module ($- \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ sends injections to injections)

Basic facts about flat modules:

- M free R-module $\Rightarrow$ M is flat
- $\mathbb{Z}$-module is flat iff torsionfree

($- \otimes \mathbb{Z}/n$ sends $\mathbb{Z} \xhookrightarrow{\cdot n} \mathbb{Z}$ to $\mathbb{Z}/n \xrightarrow{\cdot 0} \mathbb{Z}/n$ - NOT injection)

- R local, M finite R-module $\Rightarrow$ M flat iff free
  $\left(M = \sum_{i=1} R \cdot m_i\right)$
- localizations of flat morphisms are flat

Exercise: $\varphi: A \to B$ flat ring hom $\Leftrightarrow$ $\varphi^\#: \operatorname{Spec} B \to \operatorname{Spec} A$ is flat

Ex: 1) open immersions are flat; closed immersions aren't

2) $\operatorname{Spec} k[x]/_{x^2} \to \operatorname{Spec} k$ is not flat

Intuition (not obvious, sorry!):

$f: X \to Y$ flat means that

„fibers of f vary in a controlled way"

That is, flatness is a weaker property than requiring all fibers to be $\cong$, but it allows to „control" the differencies.

Thm. X, Y (loc) Noetherian, $f: X \to Y$ flat $\Rightarrow$

$$\dim_x f^{-1}(y) = \dim_x X - \dim_y Y, \quad y = f(x)$$

fiber over y

max length of chains of irred closed subsets $\{x\} \subseteq Z_0 \subsetneq Z_1 \dots \subsetneq Z_d \subseteq U$ minimizing over open $x \in U \subseteq X$

Ex: $\dim_x \mathbb{A}^2 = 2 \quad \forall x$ because

$$\underset{Z_0}{\{\text{point}\}} \subset \underset{Z_1}{\text{line}} \subset \underset{Z_2}{\text{plane}}$$

non-examinable

Cor. „Blow-up" in a closed point is not flat, because it is a map that has one fiber of different dimensions than others: that fiber gets „blown up" into a proj space while on the open complement the map is isom.

Non-examinable:

flatness is part of further interesting properties of morphisms, such as:

- $f: X \to Y$ is smooth (Jacobians don't vanish) $\Rightarrow$ f is flat

- $f: X \to Y$ is étale (smooth & relative dimension 0) iff flat & unramified

($\rightsquigarrow$ étale cohomology which gave the proof of Weil conjectures!)