B8.3: Mathematical Modelling of Financial Derivatives —Exercises—

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Exercise Sheet 3

Part A

1. The European asset-or-nothing call pays S if $S > K$ at expiry, and nothing if $S \leq K$. What is its value?

2. What is the probability that a European call will expire in the money?

Part B

1. In the Black–Scholes model, show that the value of European call option on an asset that pays a constant continuous dividend yield (i.e., $S\,D\,dt$ as in the lectures) lies below the payoff for large enough values of S. Show also that the call on an asset with dividends is less valuable than the call on an asset without dividends.

2. What is the random walk followed by the the forward price $F(S,t) = S\,e^{r\,(T-t)}$ in the Black-Scholes model?

3. Suppose that $V(S,t)$ satisfies the Black–Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)\,S\,\frac{\partial V}{\partial S} - r\,V = 0, \quad S > 0,\ t < T,$$
$$V(S,T) = P_{\mathrm{o}}(S), \quad S > 0.$$

Use the chain rule to show that if $F = S\,e^{(r-q)(T-t)}$ (the forward price of S over the time interval $[t,T]$), $t' = t$ and $\hat{V}(F,t') = V(S,t)$ then

$$\frac{\partial \hat{V}}{\partial t'} + \tfrac{1}{2}\sigma^2 F^2 \frac{\partial^2 \hat{V}}{\partial F^2} - r\,\hat{V} = 0, \quad F > 0,\ t' < T,$$
$$\hat{V}(F,T) = P_{\mathrm{o}}(F), \quad F > 0.$$

Part C

1. Consider the following perpetual American option problem. The option's payoff is

$$P_{\mathrm{o}}(S) = \begin{cases} K - S/3 & \text{if } 0 < S \le K, \\ 0 & \text{if } S > K. \end{cases}$$

Assume that the option value satisfies the steady-state Black–Scholes equation

$$\mathcal{L}_{\mathrm{ssbs}}[V] = \tfrac{1}{2}\sigma^2 S^2 V''(S) + (r-q)\, S\, V'(S) - r\, V = 0, \quad \hat{S} < S,$$

where $0 < \hat{S} \le K$ is the optimal exercise boundary and where $\sigma > 0$, $r > 0$ and $q > 0$ are constants. The option satisfies the boundary conditions

$$V(\hat{S}) = K - \hat{S}/3, \qquad \lim_{S\to\infty} V(S) \to 0.$$

(a) Give a *sketch* of the payoff and option price as functions of S and indicate where $\mathcal{L}_{\mathrm{ssbs}}[V] = 0$, where $\mathcal{L}_{\mathrm{ssbs}}[V] < 0$, where $V(S) > P_{\mathrm{o}}(S)$ and where $V(S) = P_{\mathrm{o}}(S)$.

(b) Show that, under the assumptions given above, the quadratic

$$p(m) = \tfrac{1}{2}\,\sigma^2\, m\,(m-1) + (r-q)\, m - r$$

has two distinct real roots and only one of these is strictly negative.

(c) Assume that we have smooth pasting at $\hat{S}$, i.e., $V'(\hat{S}) = -1/3$. Show that this implies that

$$\hat{S} = \frac{3m^-}{m^- - 1}\, K,$$

where $m^- < 0$ is the negative root of the quadratic $p(m)$.

(d) Show that smooth pasting only makes sense if $-\frac{1}{2} < m^- < 0$.

(e) What is the optimal exercise boundary if $m^- < -\frac{1}{2}$? Justify your answer.

(f) Suppose that $-\frac{1}{2} < m^- < 0$, so that smooth pasting does give the correct optimal exercise boundary. Suppose also that the holder of the option decides that they are going to ignore the optimal exercise boundary $\hat{S}$ and simply exercise the option as soon as $S \le \bar{S}$ where $0 < \bar{S} < K$ is chosen by the holder. In this case the value of the option, $\bar{V}(S,t)$, satisfies the problem

$$\mathcal{L}_{\mathrm{ssbs}}[\,\bar{V}\,] = 0, \quad S > \bar{S},$$
$$\bar{V}(\bar{S}) = K - \bar{S}/3, \quad \lim_{S\to\infty} \bar{V}(S) \to 0.$$

Find $\bar{V}(S)$ and show that

i. if $\bar{S} > \hat{S}$ then one could increase the value of the option by decreasing $\bar{S}$ (hint; differentiate with respect to $\bar{S}$);

ii. if $\bar{S} < \hat{S}$ then there is a potential arbitrage in the price $\bar{V}(S)$ (hint; differentiate with respect to S).