Geometric Group Theory

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This week's quotations

G.H. Hardy: "A person's first duty, a young person's at any rate, is to be ambitious, and the noblest ambition is that of leaving behind something of permanent value."

Aristotle in "Metaphysics":

"The chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree."

Graphs of groups

Definition

Let Y be an oriented graph such that the corresponding unoriented graph is connected and each of its edges appears with both orientations in Y.

A graph of groups is a pair (G, Y), where G is a map that assigns a group G_v to each vertex $v \in V(Y)$ and a group G_e to each edge $e \in E(Y)$ such that

1. $G_e = G_{\bar{e}}$
2. for all edges e, there exists an injective homomorphism $\alpha_e : G_e \to G_{t(e)}$

where $t(e)$ is the terminus of the edge $e = [o(e), t(e)]$.

Graphs of groups

Definition

The path group of a graph of groups (G, Y) is

$$F(G,Y) = \langle \bigcup_{v \in V} G_v \cup E(Y) | \bar{e} = e^{-1}, e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g), \forall e \in E(Y), g \in G_e \rangle.$$

If $G_v = \langle S_v | R_v \rangle$ then

$$F(G,Y) = \langle \bigcup_{v \in V} S_v \cup E(Y) | \bigcup_{v \in V(Y)} R_v, \bar{e} = e^{-1}, e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g) \rangle.$$

NB The group we are interested in will be defined first as a subgroup, then as a quotient of $F(G, Y)$.

Definition

A path in (G, Y) is a sequence

$$c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$$

such that $t(e_i) = o(e_{i+1})$ and $g_i \in G_{t(e_i)} = G_{o(e_{i+1})}$. We call this a path from $v_0 = o(e_1)$ to $v_n = t(e_n)$. We call

$$v_0, v_1 = t(e_1) = o(e_2), ..., v_i = t(e_i) = o(e_{i+1}), ..., v_n$$

its sequence of vertices. We define $|c|$ to be the element of the path group $g_0 e_1 g_1 ... e_n g_n$. If $a_0, a_1 \in V(Y)$ then we define

$$\pi[a_0, a_1] = \{|c| : c \text{ a path from } a_0 \text{ to } a_1\}.$$

Fundamental group of (G, Y) wrto a basepoint

Remark

If $a_0, a_1, a_2 \in V(Y)$ and $\gamma \in \pi[a_0, a_1]$, $\delta \in \pi[a_1, a_2]$ then $\gamma\delta \in \pi[a_0, a_2]$.

Proposition

Let (G, Y) be a graph of groups and suppose $a_0 \in V(Y)$. The set $\pi[a_0, a_0]$ is a subgroup of $F(G, Y)$.

We call this subgroup the fundamental group of the graph of groups (G, Y) with basepoint a_0 and denote it $\pi_1(G, Y, a_0)$.

This is the definition as a subgroup. Problem: it seems to depend on a_0.

Fundamental group of (G, Y) wrto a maximal subtree

Definition

Let (G, Y) be a graph of groups, and let T be a maximal subtree of Y. The fundamental group of (G, Y) with respect to T, denoted $\pi_1(G, Y, T)$, is

$$F(G, Y)/\langle\langle\{e : e \in T\}\rangle\rangle$$

This is the definition as a quotient of $F(G, Y)$. Problems: it seems to depend on T. Why isometric to the first?

Let $q : F(G, Y) \to \pi_1(G, Y, T)$ be the quotient map.

Proposition

$q\big|_{\pi_1(G,Y,a_0)}$ *is an isomorphism to* $\pi_1(G, Y, T)$.

Graphs of groups

Proposition

$q\big|_{\pi_1(G,Y,a_0)}$ *is an isomorphism to* $\pi_1(G, Y, T)$.

Proof: We define a homomorphism $f : \pi_1(G, Y, T) \to \pi_1(G, Y, a_0)$ as follows.

$\forall a \in V(Y)$, there exists a unique geodesic path $e_1, e_2, ..., e_n$ in T from a_0 to a. Set

$$g_a := e_1 ... e_n \in F(G, Y)$$
$$g_{a_0} := 1$$

We first define $\hat{f} : F(G, Y) \to \pi_1(G, Y, a_0)$:

- $\forall g \in G_a$, set $\hat{f}(g) = g_a g g_a^{-1} \in \pi_1(G, Y, a_0)$
- $\forall e \in E^+(Y)$ with $o(e) = a, t(e) = b$, set $\hat{f}(e) = g_a e g_b^{-1} \in \pi_1(G, Y, a_0)$.

The definition of $\hat{f}$ is consistent with the relations:

1. $\bar{e} = e^{-1}$ since for $o(e) = a, t(e) = b$,

$$\hat{f}(\bar{e}) = g_b \bar{e} g_a^{-1} = \left(g_a e g_b^{-1}\right)^{-1} = \hat{f}(e)^{-1}.$$

2. $e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g)$ since if $e = [P, Q]$:

$$\begin{aligned} \hat{f}(e\alpha_e(g)\bar{e}) &= (g_P e g_Q^{-1})(g_Q \alpha_e(g) g_Q^{-1})(g_Q \bar{e} g_P^{-1}) \\ &= g_P(e\alpha_e(g)e^{-1})g_P^{-1} \\ &= g_P \alpha_{\bar{e}}(g) g_P^{-1} \end{aligned}$$

So $\hat{f}$ is defined on $F(G, Y)$.

For every $e = [P, Q] \in T$, $\hat{f}(e) = g_P e g_Q^{-1} = 1$.

Hence $\hat{f}$ defines $f : \pi_1(G, Y, T) \to \pi_1(G, Y, a_0)$.

Recall that q is the restriction to $\pi_1(G, Y, a_0)$ of the quotient map $F(G, Y) \to \pi_1(G, Y, T)$.

Consider $q \circ f : \pi_1(G, Y, T) \to \pi_1(G, Y, T)$. For all $g \in G_a$,

$$q \circ f(g) = q(g_a g g_a^{-1}) = g.$$

For all $e \notin T$,

$$q \circ f(e) = q(g_P e g_Q^{-1}) = e.$$

Hence $q \circ f = \mathrm{id}$.

Graphs of groups

Consider now $f \circ q : \pi_1(G, Y, a_0) \to \pi_1(G, Y, a_0)$.

If $g_0 e_1 ... e_n g_n$ arbitrary in $\pi_1(G, Y, a_0)$ and $e_i = [P_{i-1}, P_i]$,

$$
\begin{aligned}
f \circ q(g_0 e_1 ... e_n g_n) &= g_0 (e_1 g_{P_1}^{-1})(g_{P_1} g_1 g_{P_1}^{-1})(g_{P_1} e_2 g_{P_2}^{-1}) ... g_{P_{n-1}}^{-1}(g_{P_{n-1}} e_n) g_n \\
&= g_0 e_1 ... e_n g_n
\end{aligned}
$$

NB If $e_i \in T$ then $f \circ q(g_{i-1} e_i g_i) = f(g_{i-1} g_i) = g_{P_{i-1}} g_{i-1} g_{P_{i-1}}^{-1} g_{P_i} g_i g_{P_i}^{-1} = g_{P_{i-1}} g_{i-1} e_i g_i g_{P_i}^{-1}$. □

Proposition

$q\big|_{\pi_1(G,Y,a_0)}$ is an isomorphism to $\pi_1(G, Y, T)$.

Corollary

The fundamental group $\pi_1(G, Y, a_0)$ of the graph of groups (G, Y) does not depend on the choice of basepoint a_0.

Corollary

The quotient $\pi_1(G, Y, T)$ of the path group does not depend on the choice of the tree T.

Reduced paths of graphs of groups

Definition

Let (G, Y) be a graph of groups. A path $c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$ is reduced if

1. $g_0 \neq 1$ if $n = 0$;
2. If $e_{i+1} = \bar{e}_i$ then $g_i \notin \alpha_{e_i}(G_{e_i})$.

We say that $g_0 e_1 ... e_n g_n$ is a reduced word.

Recall that $|c|$ is the element in $F(G, Y)$ represented by a path c.

Theorem

If c is a reduced path then $|c| \neq 1$ in $F(G, Y)$. In particular, $G_v \hookrightarrow F(G, Y)$ is injective for every $v \in V(Y)$.

Reduced paths of graphs of groups

Theorem

If c is a reduced path then $|c| \neq 1$ in $F(G, Y)$. In particular, $G_v \hookrightarrow F(G, Y)$ is injective for every $v \in V(Y)$.

Proof

First assume that Y is finite. We will argue by induction on the number of edges in Y. If there are no edges, then the theorem holds. So assume the theorem is true for graphs with n edges, and suppose that Y has $n+1$ edges.

Reduced paths of graphs of groups

Case 1: $Y = Y' \cup \{e\}$, $o(e) \in V(Y')$, $v = t(e) \notin V(Y')$. Then

$$F(G, Y) = (F(G, Y') * G_v)*_{\alpha_e(G_e)}$$

with stable letter e. A reduced word containing e corresponds to a reduced word in the HNN extension that is $\neq 1$.

Case 2: $Y = Y' \cup \{e\}$, $\{o(e), t(e)\} \subseteq V(Y')$. Then

$$F(G, Y) = F(G, Y')*_{\alpha_e(G_e)}$$

and the comment above applies again.

Now suppose that Y is infinite. Any reduced path c involves finitely many orbits of vertices and edges and so c lies within a finite subgraph Y_1 of Y.

c is a reduced path in $F(G, Y_1)$ and so $c \neq 1$ in $F(G, Y_1)$. □

Reduced paths of graphs of groups

Theorem

If c is a reduced path then $|c| \neq 1$ in $F(G, Y)$. In particular, $G_v \hookrightarrow F(G, Y)$ is injective for every $v \in V(Y)$.

Corollary

For every $v \in V(Y)$, the homomorphism $G_v \to \pi_1(G, Y, T)$ is injective.

Proof.

$G_v \to F(G, Y)$ is injective and $\pi : \pi_1(G, Y, v) \to \pi_1(G, Y, T)$ is an isomorphism. □

Graphs of groups

One can easily see that

1. If Y has 2 vertices and one edge then
$$\pi_1(G, Y, T) = G_u *_{G_e} G_v .$$
2. If Y has 1 vertex and 1 edge with stable letter 'e' then
$$\pi_1(G, Y, T) = G_v *_{\alpha_e(G_e)}$$
and $\theta : \alpha_e(G_e) \to \alpha_{\bar{e}}(G_e) \in G_v$, $\theta(g) = \alpha_{\bar{e}} \circ \alpha_e^{-1}$.
3. If $Y = Y' \cup \{e\}$ and $t(e) = v \notin Y'$ then
$$\pi_1(G, Y, T) = \pi_1(G, Y', T') *_{G_e} G_v .$$
4. If $Y = Y' \cup \{e\}$ and $v = t(e) \in Y'$ then
$$\pi_1(G, Y, T) = \pi_1(G, Y', T) *_{\alpha_e(G_e)} .$$