Chapter 8
Sheaves of modules

§ $\mathcal{O}_X$-modules

def. $(X, \mathcal{O}_X)$ ringed space. A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ of ab groups s.t. for $U \subseteq X$ open there is a multiplication

$$\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U),$$

compatible with restriction maps.

A sheaf of $\mathcal{O}_X$-algebras: similar def (groups $\rightsquigarrow$ rings)

Fact Sheaves of $\mathcal{O}_X$-modules ($\mathcal{O}_X$-algebras) form an abelian category $\mathcal{O}_X$-Mod: Ker; Im; Coker; $\oplus$; Π; $\subseteq$; $\otimes$; $\mathcal{H}om$

apply on each U, then sheafify!

Denote $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}')$: $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ – sheaf of $\mathcal{O}_X$-mod

Rem: $\mathcal{F}$ $\mathcal{O}_X$-module $\Rightarrow$ $\mathcal{F}_x$ an $\mathcal{O}_{X,x}$-module, and $\mathcal{F} \to \mathcal{F}'$ induces $\mathcal{F}_x \to \mathcal{F}'_x$ map of $\mathcal{O}_{X,x}$-modules.

Example: (more later!)
$X = \mathbb{P}^n_{\mathbb{C}}$ the variety

$$\mathcal{O}_{\mathbb{P}^n}(d): \quad U \mapsto \left\{ \frac{P(x_0, \dots, x_n)}{Q(x_0, \dots, x_n)} \text{ rational homogeneous functions of deg } d \text{ regular at all pts of } U \right\}$$

in particular, $\mathcal{O}_{\mathbb{P}^n}(d)(\mathbb{P}^n) = $ homog. poly of deg d in $x_0, \dots, x_n$

Since $\mathcal{O}_{\mathbb{P}^n}(U)$ consists of ratios of polynomials of same degree, we have multiplication:

$$\mathcal{O}_{\mathbb{P}^n}(U) \times \mathcal{O}_{\mathbb{P}^n}(d)(U) \xrightarrow{\cdot} \mathcal{O}_{\mathbb{P}^n}(d)(U)$$

Rem: $d<0 \rightsquigarrow \mathcal{O}_{\mathbb{P}^n}(d)$ has no global sections, but still makes sense! and is interesting :)

Moving between spaces

$f: X \to Y$ ringed spaces

$\mathcal{F}$ sheaf of $\mathcal{O}_X$-modules [abbreviate: $\mathcal{O}_X$-module]

↳ $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module

↳ $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module via $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$

Conversely:

$\mathcal{G}$ an $\mathcal{O}_Y$-module

↳ $f^{-1}\mathcal{G}$ an $f^{-1}\mathcal{O}_Y$-module

↳ $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ an $\mathcal{O}_X$-module

via $f^\#$

Claim: (f^*, f_*) are adjoint functors for modules over ringed spaces, i.e. $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$ $\forall \mathcal{F}, \mathcal{G}$ as above.

Rem. $f: X \to Y$ flat $\Rightarrow f^*: \mathcal{O}_Y\text{-Mod} \to \mathcal{O}_X\text{-Mod}$ is exact (preserves s.e.s.), because so do f^{-1} and $- \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X$ ← flat

§ (Quasi-) coherent sheaves

def. R ring, M R-module, $X = \operatorname{Spec} R$.

The sheaf associated to M is

$$\tilde{M}: \quad D(f) \mapsto M_f,$$

extended to a sheaf on $\operatorname{Spec} R$ in the same way as we defined $\mathcal{O}_X$.

In particular,

$$\tilde{M}(X) = M; \quad \tilde{M}_p = M_p; \quad \tilde{R} = \mathcal{O}_X,$$

and $\tilde{M}$ is the sheafification of

$$U \mapsto M \underset{R}{\otimes} \mathcal{O}_X(U).$$

this allows to define $\tilde{M}$ for any scheme $Y \to \operatorname{Spec} R$!

Let now X be any scheme.

def. A quasi-coherent sheaf $\mathcal{F}$ on X is an $\mathcal{O}_X$-module s.t. $\exists$ affine open cover $X = \bigcup U_i$ where $\mathcal{F}|_{U_i} \overset{\varphi_i}{\underset{\sim}{\to}} \tilde{M}_i$ for some M_i modules over $\mathcal{O}_X(U_i)$, and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on U_{ijk}.

$\mathcal{F}$ is coherent if M_i are fin. gen. modules (this notion behaves well for Noetherian X, for non-Noetherian X the right def is more technical!)

Ex. $\mathcal{O}_X^{\oplus n}$ is coherent for X Noeth

Rethinking closed immersions

Recall: $i : Z \hookrightarrow X$ homeo onto a closed subset and $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.

Denote $\mathcal{I}_{Z/X} := \ker i^\#$.

Lemma (easy)

1) $\mathcal{I}_{Z/X}$ is a sheaf of ideals on X, i.e. $\mathcal{I}_{Z/X}(U)$ is an ideal in $\mathcal{O}(U)$ $\forall U \subseteq X$ open.

2) $\mathcal{I}_{Z/X}$ is qcoh; coh when X is Noetherian

3) there's a bijection between qcoh sheaves of ideals of X and closed subschemes of X.

3) can be used for an alternative def of closed subschemes

Criterion: An $\mathcal{O}_X$-module $\mathcal{F}$ is qcoh iff $\forall\, U = \operatorname{Spec} R \subseteq X$ open $\mathcal{F}|_U$ is a sheaf assoc. to an R-module M.

If X Noetherian, then $\mathcal{F}$ is coherent iff M's are fin. generated.

Quasi-coherence is a local property.

Cor. X affine $\Rightarrow$

$$\mathrm{QCoh}(X) \simeq \mathrm{Mod}_{\mathcal{O}_X(X)}$$

$$\mathcal{F} \longmapsto \mathcal{F}(X)$$

$$\tilde{M} \longleftarrow M$$

Ex: not every $\mathcal{O}_X$-module is quasi-coherent (although in practice they often are)

1) $X = \operatorname{Spec} k[x]_{(x)} = \{ m, \eta \}$

$\mathcal{F}(X) := 0$, $\mathcal{F}(\eta) := k(x)$ — it's a sheaf of $\mathcal{O}_X$-modules which is not quasi-coherent, otherwise we'd have $\mathcal{F}(X) = 0 \Rightarrow \mathcal{F}(\eta) = 0$.

2) $X = \operatorname{Spec} k[t]$, $\mathcal{F}(U) := \begin{cases} \mathcal{O}_X(U) & \text{if } \{0\} \notin U \\ 0 & \text{if } \{0\} \in U \end{cases}$

it's not qcoh because $\mathcal{F}(X) = 0$.

3) scyscraper sheaf is not qcoh:

$X = \mathbb{A}^1_k$, $\mathcal{F}(U) = \begin{cases} k[x] & \text{if } 0 \in U \\ 0 & \text{else} \end{cases}$

$\mathcal{F} \neq \tilde{M}$ because $k[x] = \mathcal{O}_X \not\cong \mathcal{F}$.

§ Properties of (q)coh sheaves

Prop. 1) X Noetherian, $f: \mathcal{F} \to \mathcal{G}$ (q)coh $\mathcal{O}_X$-modules. Then $\ker f$, $\operatorname{coker} f$, $\operatorname{im} f$ are also (q)coh.

2) $f: X \to Y \in \mathrm{Sch}$, $\mathcal{F}$ (q)coh $\mathcal{O}_Y$-module $\Rightarrow f^*\mathcal{F}$ is also (q)coh. (for coh need X Noeth. Because $f^*\mathcal{O}_Y = \mathcal{O}_X$!)

3) $\mathcal{G}$ (q)coh on X $\not\Rightarrow$ $f_*\mathcal{G}$ (q)coh on Y in general.
(although $f_*: QCoh(X) \to QCoh(Y)$ when X quasicompact & separated)

Non-Ex: 1) $f: \coprod_{n\in\mathbb{N}} \mathbb{A}^1 \to \mathbb{A}^1$; let $\mathcal{F} = \widetilde{\prod k[t]}$.

if $f_*\mathcal{F} \in QCoh(\mathbb{A}^1)$, then $f_*\mathcal{F} = \widetilde{\prod k[t]}$ because $\mathbb{A}^1$ is affine.
But: $\left(\frac{1}{t^n}\right)_n \in \mathcal{F}(\coprod D(t)) = f_*\mathcal{F}(D(t))$ yet $\left(\frac{1}{t^n}\right)_n \notin \widetilde{\prod k[t]}(D(t))$
$= (\prod k[t])_t$
$\neq \prod k[t]_t$

Hence $f_*\mathcal{F}$ is not qcoh.

2) $f: \mathbb{A}^1_k \to \operatorname{Spec} k$
$f_*\mathcal{O}_{\mathbb{A}^1_k} = k[x] \in Mod_k$ - not a f.g. k-module, so not coherent

Thm. (without proof)
$f: X \to Y$ proper morphism, X, Y Noeth. $\Rightarrow$
$f_*: Coh(X) \to Coh(Y)$.

Ex: $Y \overset{i}{\hookrightarrow} X \Rightarrow i_*\mathcal{O}_Y$ is coherent

$X = \operatorname{Spec} R \to i_*\mathcal{O}_Y = \widetilde{R/I}$ for $Y \cong \operatorname{Spec} R/I$.

Gabriel-Rosenberg thm
Cool theorem (not examinable)
X qcompact and separated (e.g. variety) $\Rightarrow$
$QCoh(X)$ determines up to isom!
(considered as an abelian cat)

§ Vector bundles

def. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is a **vector bundle** if it is locally free:
$\forall x \in X \ \exists U(x) \subseteq X$ open: $\mathcal{F}|_{U(x)} \simeq \mathcal{O}_X^{n}$

rank is locally constant, usually take it to be constant

$\mathcal{F}$ is a **line bundle** if $n=1$.

NB: not enough to ask for $\mathcal{F}_x \simeq \mathcal{O}_{X,x}^n \ \forall x$!
Fact for coherent sheaves $\mathcal{F}$ that's enough :)

Construction:

$\mathcal{F}$ can be encoded by the data $X = \bigcup U_i$ and

$$\begin{array}{ccc}
\mathcal{F}|_{U_{ij}} & \xrightarrow[\varphi_i]{\sim} & \mathcal{O}_{U_{ij}}^{n_i} \\
\shortparallel & & \downarrow \alpha_{ij} \\
\mathcal{F}|_{U_{ji}} & \xrightarrow[\varphi_j]{\sim} & \mathcal{O}_{U_{ji}}^{n_j} = \mathcal{O}_{U_{ij}}^{n_j}
\end{array}$$

α_{ij} — transition maps that satisfy $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on U_{ijk}.

Big picture:

$$\mathrm{Vect}(X) \subseteq \mathrm{Coh}(X) \subseteq \mathrm{QCoh}(X) \subseteq \mathcal{O}_X\text{-Mod}$$

nicest objects, but not an abelian category!
Ker, Coker may be not loc-free :(

Ex: 1) $X = \operatorname{Spec} R$

$\mathcal{F} = \widetilde{M}$ where M is a f.g. projective R-module

flat $\wr$ projective $\wr$ loc-free

2) $X = \mathbb{P}^n$

use the gluing description:

$X = \bigcup A_i$ where $A_i = \operatorname{Spec} \mathbb{Z}\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \simeq \mathbb{A}^n$

- $\mathcal{O}(1)$: line bundle with

  $\alpha_{ij} = \left(\frac{x_i}{x_j}\right)$ – transition maps

- $\mathcal{O}(d) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes d}$ ⟵ later: makes sense $\forall d \in \mathbb{Z}$, not just $d \geq 0$!

  $\alpha_{ij} = \left(\frac{x_i}{x_j}\right)^d, \quad d \in \mathbb{Z}$

Global sections: $\Gamma(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} \mathbb{Z}[x_0, \dots, x_n]_d, & d \geq 0 \\ 0 & \text{else} \end{cases}$ ⟵ homog poly deg d

(good exercise!)

Lemma. $f: X \to Y \rightsquigarrow f^*: \operatorname{Vect}(Y) \to \operatorname{Vect}(X)$

Proof sketch: $f^* \mathcal{O}_Y = \mathcal{O}_X$

f^* commutes with $\oplus$ (coproduct in $\mathcal{O}_X$-Mod)

can check locally

Thm $f: X \to Y$ finite flat $\Rightarrow f_*: \operatorname{Vect}(X) \to \operatorname{Vect}(Y)$

for affines: $f_*(\widetilde{M}) = \widetilde{M}$ ⟵ considered as a module over $\mathcal{O}_Y(Y)$

and that's when scalar restriction preserves f.g. proj modules

Rem: in general, f_* does not preserve vector bundles, e.g. when f is a closed immersion.

§ Why vector bundles are called so?

Construction

- $\mathcal{E}$ locally free $\mathcal{O}_X$-module of rank n
- $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ also loc-free of rank n
- $\mathrm{Sym}\,\mathcal{E}^\vee$ locally free sheaf of $\mathcal{O}_X$-algebras

generalizing: $V = k^n \rightsquigarrow \mathrm{Sym}\,V = k[x_1, \dots, x_n]$

Namely, $\mathrm{Sym}\,\mathcal{F} := \bigoplus_{m \geq 0} \mathcal{F}^{\otimes m} / (s \otimes t - t \otimes s)$ for all local sections s, t

- $\underline{\mathrm{Spec}}\,\mathrm{Sym}\,\mathcal{E}^\vee =: \mathrm{Tot}(\mathcal{E})$ total space of $\mathcal{E}$

relative Spec

$$\begin{array}{c} \mathrm{Tot}(\mathcal{E}) \\ \downarrow \pi \\ X \end{array}$$

is an X-scheme s.th $\pi^{-1}(x) \simeq \mathbb{A}^n_{k(x)}$ $\forall x \in X$

and locally $\mathrm{Tot}(\mathcal{E})$ looks like $\mathbb{A}^n \times U \to U$.

In particular, $\underline{\mathrm{Spec}}\,\mathrm{Sym}(\mathcal{O}_S^{\oplus n}) \simeq \mathbb{A}^n_S$

We bring a topological construction to geometry!

More precisely, $\mathcal{A}$ sheaf of $\mathcal{O}_X$-alg qcoh as an $\mathcal{O}_X$-module $\rightsquigarrow$

Define a set $\underline{\mathrm{Spec}}\,\mathcal{A} \xrightarrow{\pi} X$ with $\pi^{-1}(p) = \mathrm{Spec}(\mathcal{A}_p \otimes k(p))$.

$\forall\, U \subseteq X$ affine open $\exists$ bijection $\pi^{-1}(U) \simeq \mathrm{Spec}\,\mathcal{A}(U) \rightsquigarrow$ define topology and ring of functions on $\underline{\mathrm{Spec}}\,\mathcal{A}$ to make π a scheme map.

- sections of $\mathcal{E}$ correspond to sections $s : X \hookrightarrow \mathrm{Tot}(\mathcal{E})$!

Because: $\forall$ affine open $U \subseteq X$

$$\underbrace{\mathrm{Hom}_{\mathrm{Sch}_X}(U, \underline{\mathrm{Spec}}\,\mathrm{Sym}\,\mathcal{E}^\vee)}_{\text{sections of } \mathrm{Tot}(\mathcal{E})} \simeq \mathrm{Hom}_{\mathrm{Alg}_{\mathcal{O}_X(U)}}(\mathrm{Sym}\,\mathcal{E}^\vee(U), \mathcal{O}_X(U)) \simeq$$

$$\simeq \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_X(U)}}(\mathcal{E}^\vee(U), \mathcal{O}_X(U)) = \mathcal{E}^{\vee\vee}(U) \simeq \underbrace{\mathcal{E}(U)}_{\text{sections of } \mathcal{E}}.$$

def. An $\mathcal{O}_X$-module $\mathcal{I}$ is invertible if $\exists\ \mathcal{F} \in QCoh(X)$ s.t. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{I} \simeq \mathcal{O}_X$.

they form a group wrt the operation $- \otimes_{\mathcal{O}_X} -$

Thm. $\mathcal{I} \in \mathcal{O}_X$-Mod is invertible iff $\mathcal{I}$ is a line bundle.

Semi-Proof: ① $\mathcal{I}$ line bundle $\rightsquigarrow$ consider its dual bundle $\mathcal{I}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X)$

Claim: $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{I}^\vee \xrightarrow{\sim} \mathcal{O}_X$ canonical pairing

over affine bases where $\mathcal{I}, \mathcal{I}^\vee$ are trivial this is the canonical map $R \otimes_R Hom_R(R, R) \xrightarrow{\sim} R$

② $\mathcal{I}$ invertible $\Rightarrow$ locally on affine opens $\mathcal{I} \otimes \mathcal{F} \xrightarrow{\sim} \mathcal{O}_X$ becomes $M \otimes_R N \xrightarrow{\sim} R$.

Claim: such M and N are projective of finite rank.
exercise in comm. alg! (nothing fancy)

Then M, N are locally free and $M_p \otimes_{R_p} N_p \simeq R_p$ implies M, N are loc-free of rank 1, so $\mathcal{I}$ is a line bundle.