Further Partial Differential Equations (2024)
Problem Sheet 3

1. **Inwardly radial spreading in a porous medium**

   Consider again the radial spreading of a fixed volume of liquid in a porous medium as described by equation (10) in Problem Sheet 2. Suppose that the liquid is now confined in a cylindrical container of radius $\hat{r}_0$ and the liquid occupies a region $\hat{r}_f(\hat{t}) \leq \hat{r} \leq \hat{r}_0$ where $\hat{r}_f$ moves inwardly with time.

   (a) Write down the equation that expresses conservation of mass in this case and comment on how it differs from that in question 2.

   (b) By using the results of question 2, show that the system may be reduced to one that contains no physical parameters.

   (c) Let t_c denote the time at which the central dry hole closes. Define

   $$\tau = t_c - t, \qquad h = \frac{r^2}{\tau}\bar{h}(r,\tau), \qquad Q = \frac{r}{\tau}\bar{Q}(r,\tau) \tag{1}$$

   and show that in terms of these new variables the system may be written as

   $$2\bar{h} + \bar{Q} + r\frac{\partial \bar{h}}{\partial r} = 0, \tag{2}$$

   $$\tau\frac{\partial \bar{h}}{\partial \tau} - \bar{h} - 4\bar{h}\bar{Q} - r\frac{\partial}{\partial r}\left(\bar{h}\bar{Q}\right) = 0. \tag{3}$$

   (d) Now suppose that $\bar{h} = \bar{h}(\eta)$, $\bar{Q} = \bar{Q}(\eta)$ where $\eta = r/\tau^\alpha$ is a similarity variable, for some α. Find the equations that are satisfied by $\bar{h}$ and $\bar{Q}$.

   (e) Show that the system can be written in the form

   $$\frac{\mathrm{d}\bar{Q}}{\mathrm{d}\bar{h}} = \frac{\bar{h} + 4\bar{h}\bar{Q} - \alpha(\bar{Q} + 2\bar{h}) - \bar{Q}(\bar{Q} + 2\bar{h})}{\bar{h}(\bar{Q} + 2\bar{h})}. \tag{4}$$

   (f) Based on the results of this analysis, is this solution a similarity solution of the first or second kind? What physical feature of this problem indicates that it is a similarity solution of this kind?

Solution

(a) Conservation of mass is expressed via the equation

$$2\pi \int_{\hat{r}_f}^{\hat{r}_0} \hat{r}\hat{h}(\hat{r},\hat{t})\,\mathrm{d}\hat{r} = \hat{V}, \tag{5}$$

where $\hat{V}$ is the volume of liquid. This differs from the result in question 2(a) since now we have an additional length scale in the problem: the radius of the cylinder.

(b) We non-dimensionalize in exactly the same way as in question 2(b) except now the radial scaling $\hat{r}_0$ that was arbitrary in problem 1 is now chosen to be the radius of the cylinder. The governing equations and mass conservation then become

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rhQ) = 0, \tag{6}$$

$$Q = -\frac{\partial h}{\partial r}, \tag{7}$$

$$\int_{r_f}^{1} rh\,\mathrm{d}r = 1. \tag{8}$$

(c) This is a straightforward substitution.

(d) Making the substitution proposed gives

$$2\bar{h} + \bar{Q} + \eta\bar{h}' = 0, \tag{9}$$

$$-\alpha\eta\bar{h}' - \bar{h} - 4\bar{h}\bar{Q} - \eta\left(\bar{h}\bar{Q}\right)' = 0. \tag{10}$$

(e) Solving for $\bar{Q}'$ and $\bar{h}'$ and dividing one by the other gives the required result.

(f) This is a similarity solution of the second kind as we have not yet found the value of the scaling. This comes from the application of boundary conditions. Note that in the original formulation, we have a natural lengthscale, which is the radius of the container, $\hat{r}_0$. This is what prohibits a similarity solution of the first kind. However, with the introduction of the variable $\eta = r/\tau^\alpha$, since $\tau \to 0$ as the hole closes up, $\eta \to \infty$ and so we lose the lengthscale, enabling a similarity solution. However, the price we pay is that η_f cannot be found from our similarity analysis: this would need to be determined by comparing our similarity solution with the actual numerical (or experimental) solution at a particular point in time.

We find three fixed points of the system (4): $(\bar{h},\bar{Q}) = (0,0)$, $(0,-\alpha)$ and $(1/8,-1/4)$. The first point is the trivial solution. The second fixed point corresponds to the moving front and the third point corresponds to the fluid arriving at the origin. If we apply the first and second conditions then this provides two boundary conditions for the first-order system, which forms an eigenvalue problem. The eigenvalue is found to be $\alpha \approx 0.856$.

2. **Asymptotic analysis of Stefan problems**

   Show that the transcendental relation (2.12) between β and St may be parameterized as

$$\mathrm{St} = \sqrt{\pi}\xi \mathrm{e}^{\xi^2} \mathrm{erf}(\xi), \qquad \beta = \frac{2\sqrt{\xi}\mathrm{e}^{-\xi^2/2}}{\pi^{1/4}\sqrt{\mathrm{erf}(\xi)}}, \tag{11}$$

   where $0 < \xi < \infty$. By taking the limits $\xi \to 0$ and $\xi \to \infty$, derive the asymptotic expressions (2.13).

Solution

Define $\xi = \beta\sqrt{\mathrm{St}}/2$. Substituting into the transcendental equation (2.12) gives

$$\sqrt{\pi}\xi \mathrm{e}^{\xi^2}\operatorname{erf}(\xi) = \mathrm{St}, \tag{12}$$

$$\beta = \frac{2\xi}{\mathrm{St}} = \frac{2\sqrt{\xi}\mathrm{e}^{-\xi^2/2}}{\pi^{1/4}\sqrt{\operatorname{erf}(\xi)}}. \tag{13}$$

As $\xi \to 0$, $\operatorname{erf}(\xi) \sim 2\xi/\sqrt{\pi}$ so $\mathrm{St} \sim 2\xi^2$ in (12) and $\beta \to \sqrt{2}$ in (13).

As $\xi \to \infty$, $\operatorname{erf} \to 1$, so (12) and (13) give respectively

$$\mathrm{St} \sim \sqrt{\pi}\xi \mathrm{e}^{\xi^2}, \tag{14}$$

$$\beta \sim \frac{2}{\pi^{1/4}}\xi^{1/2}\mathrm{e}^{-\xi^2/2}. \tag{15}$$

Equation (14) gives

$$\log\left(\frac{\mathrm{St}}{\sqrt{\pi}}\right) \sim \xi^2 + \text{higher order logarithmic terms}, \tag{16}$$

and so

$$\beta \sim \frac{2}{\sqrt{\mathrm{St}}}\sqrt{\log\left(\frac{\mathrm{St}}{\sqrt{\pi}}\right)}. \tag{17}$$

3. **Similarity solutions in the two-phase Stefan problem**

Consider the two-phase Stefan problem (2.15) in the limit $t \to 0$. Show that the leading-order behaviour is given by

$$u(x,t) \sim \begin{cases} f(\eta) & 0 < \eta < \beta, \\ g(\eta) & \beta < \eta < \infty, \end{cases} \qquad s(t) \sim \beta\sqrt{t}, \qquad \eta = \frac{x}{\sqrt{t}},$$

where

$$g(\eta) = \theta\left(\frac{\operatorname{erfc}\left(\eta\sqrt{\mathrm{St}}/2\sqrt{\kappa}\right)}{\operatorname{erfc}\left(\beta\sqrt{\mathrm{St}}/2\sqrt{\kappa}\right)} - 1\right), \qquad f(\eta) = \left(1 - \frac{\operatorname{erf}\left(\eta\sqrt{\mathrm{St}}/2\right)}{\operatorname{erf}\left(\beta\sqrt{\mathrm{St}}/2\right)}\right),$$

and β satisfies the transcendental equation

$$\frac{\beta\sqrt{\pi}}{2\sqrt{\mathrm{St}}} = \frac{\mathrm{e}^{-\beta^2\mathrm{St}/4}}{\operatorname{erf}\left(\beta\sqrt{\mathrm{St}}/2\right)} - \frac{K\theta\mathrm{e}^{-\beta^2\mathrm{St}/4\kappa}}{\sqrt{\kappa}\operatorname{erfc}\left(\beta\sqrt{\mathrm{St}}/2\sqrt{\kappa}\right)}.$$

Solution

Substitute in the similarity solution form given in the question. (Note that you can obtain the form of this similarity solution by using a scaling argument.) This transforms the problem to

$$
\begin{aligned}
f'' + \frac{\mathrm{St}}{2}\eta f' &= 0, && \eta < \beta, && (18)\\
g'' + \frac{\mathrm{St}}{2\kappa}\eta g' &= 0, && \eta > \beta, && (19)\\
f(0) &= 1, && && (20)\\
g &\to -\theta && \text{as} \quad \eta \to \infty, && (21)\\
f(\beta) &= g(\beta) = 0, && && (22)\\
Kg'(\beta) - f'(\beta) &= \frac{\beta}{2}. && && (23)
\end{aligned}
$$

The solution follows straightforwardly from this.