Geometric Group Theory

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Graphs of groups

We can check that

1. If Y has 2 vertices and one edge then

$$\pi_1(G, Y, T) = G_u *_{G_e} G_v .$$

2. If Y has 1 vertex and 1 edge with stable letter 'e' then

$$\pi_1(G, Y, T) = G_v *_{\alpha_e(G_e)}$$

and $\theta : \alpha_e(G_e) \to \alpha_{\bar{e}}(G_e) \in G_v$, $\theta(g) = \alpha_{\bar{e}} \circ \alpha_e^{-1}$.

3. If $Y = Y' \cup \{e\}$ and $t(e) = v \notin Y'$ then

$$\pi_1(G, Y, T) = \pi_1(G, Y', T') *_{G_e} G_v .$$

4. If $Y = Y' \cup \{e\}$ and $v = t(e) \in Y'$ then

$$\pi_1(G, Y, T) = \pi_1(G, Y', T) *_{\alpha_e(G_e)} .$$

Reduced words of graphs of groups

We will find a choice of representatives for elements in $F(G, Y)$, where (G, Y) is a graph of groups. For each edge $e \in E(Y)$, pick a set S_e of left coset representatives of $\alpha_{\bar{e}}(G_e)$ in $G_{o(e)}$, with $1 \in S_e$.

Definition

An S-reduced path is a path $(s_1, e_1, ..., s_n, e_n, g)$ with

- $s_i \in S_{e_i} \ \forall i$;
- $s_i \neq 1$ if $e_i = \bar{e}_{i-1}$;
- $g \in G_{t(e_n)}$.

Lemma

Given $a, b \in V(Y)$, every element in $\pi[a, b]$ is represented by a unique S-reduced path.

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Proof

Existence: Let $\gamma \in \pi[a, b]$ and consider the path $c = (g_0, e_1, g_1, e_2, \dots, g_{n-1}, e_n, g_n)$ such that $t(e_i) = o(e_{i+1})$, $g_i \in G_{t(e_i)} = G_{o(e_{i+1})}$ and $\gamma = |c|$.

We will prove by induction on n that γ can be represented by an S-reduced path. For $n = 0$ it is obvious. For $n = 1$,

$$\gamma = g_0 e_1 g_1 = s_0 \alpha_{\bar{e}_1}(h_0) e_1 g_1 = s_0 e_1 \alpha_{e_1}(h_0) g_1 = s_0 e_1 g_1'$$

A similar argument holds for the inductive step.

Reduced words of graphs of groups

Uniqueness: Consider two reduced paths

$$c = (s_1, e_1, ..., s_n, e_n, g)$$
$$c' = (\sigma_1, \eta_1, ..., \sigma_k, \eta_k, \gamma)$$

such that $|c| = |c'|$. Then

$$\gamma^{-1}\eta_k^{-1}\sigma_k^{-1}...\eta_1^{-1}\sigma_1^{-1}s_1e_1...s_ne_ng = 1$$

We will prove that $c = c'$ by induction on the length. The above word cannot be reduced hence $\eta_1^{-1} = e_1^{-1}$ and $\sigma_1^{-1}s_1 \in \alpha_{\bar{e}_1}(G_{e_1})$. So $\sigma_1 = s_1$. And so we can apply the inductive assumption. □

Graphs of groups and actions on trees

Theorem

$H = \pi_1(G, Y, a_0)$ acts on a tree T without inversions and such that

1. *The quotient graph $H\backslash T$ can be identified with Y;*
2. *Let $q : T \to Y$ be the quotient map:*
   - a. *For all $v \in V(T)$, $\mathrm{Stab}_H(v)$ is a conjugate in H of $G_{q(v)}$;*
   - b. *For all $e \in E(T)$, $\mathrm{Stab}_H(e)$ is a conjugate in H of $G_{q(e)}$.*

Proof: For all $a \in V(Y)$, we define an equivalence relation on $\pi[a_0, a]$ by

$$|c_1| \sim |c_2| \iff |c_1| = |c_2|g \text{ for some } g \in G_a$$

Vertices of the tree:

$$V(T) = \bigsqcup_{a \in V(Y)} \pi[a_0, a]/\sim$$

Graphs of groups and actions on trees

$$V(T) = \bigsqcup_{a \in V(Y)} \pi[a_0, a]/\sim$$

Every element of $\pi[a_0, a]/\sim$ has a unique representative corresponding to an S-reduced path of the form $(s_1, e_1, ..., s_n, e_n)$, $o(e_1) = a_0$, $t(e_n) = a$.
Thus $V(T)$ can also be identified with S-reduced paths as above.

Edges of the tree: $\{(s_1, e_1, ..., s_n, e_n), (s_1, e_1, ..., s_n, e_n, s_{n+1}, e_{n+1})\}$.
Connectedness is obvious.

By our definition of edges, a cycle/circuit gives an S-reduced path with corresponding element $1 \in \pi[a_0, a]$ contradicting the uniqueness of the representation of a reduced path.

Graphs of groups and actions on trees

Action of $H = \pi_1(G, Y, a_0) = \pi[a_0, a_0]$ on T: For all $h \in \pi[a_0, a_0]$ and for all $[g] \in V(T)$ (equivalence classes of $\pi[a_0, a]/\sim$) define the action

$$h \cdot [g] = [hg]$$

- $g_1 \sim g_2 \Rightarrow hg_1 \sim hg_2$ and $\{[g_1], [g_2]\}$ edge $\Rightarrow \{[hg_1], [hg_2]\}$ edge.
- If $[g_1], [g_2]$ are such that $h \cdot [g_1] = [g_2]$ then $a_1 = a_2$ where $g_i \in \pi[a_0, a_i]$.
- Conversely, if $[g_1], [g_2] \in \pi[a_0, a]$ then $h = g_2 g_1^{-1} \in \pi[a_0, a_0]$ and $h[g_1] = [g_2]$.

Thus $H \backslash V(T)$ can be identified with $V(Y)$. And likewise $H \backslash E(T)$ can be identified with $E(Y)$.

Graphs of groups and actions on trees

Stabilisers of vertices: For all $[v] \in V(T)$ with $v \in \pi[a_0, b]$, where $b \in V(Y)$,

$$h \in \mathrm{Stab}([v]) \iff hv \sim v \iff hv = vg_b \text{ for some } g_b \in G_b$$
$$\iff h = vg_b v^{-1} \text{ for some } g_b \in G_b$$

Thus $\mathrm{Stab}([v]) = vG_b v^{-1}$. This relation is in $F(G, Y)$.

Recall that each G_b was embedded in $H = \pi_1(G, Y, a_0)$ as follows:

- for a maximal subtree $T_Y \subset Y$, set $g_b = e_1 \dots e_n$ the unique geodesic path in T_Y from a_0 to b.
- $\forall g \in G_b$, identify it with $\hat{g} = g_b g g_b^{-1}$. Let $\hat{G}_b$ be the image of G_b.

The equality $\mathrm{Stab}([v]) = vG_b v^{-1}$ becomes

$\mathrm{Stab}([v]) = vg_b^{-1} \hat{G}_b g_b g = h\hat{G}_b h^{-1}$, where $h = vg_b^{-1} \in H = \pi_1(G, Y, a_0)$.

Stabilisers of edges: Every edge in $E(T)$ is of the form $\delta = [[v], [vge]]$, $v \in \pi[a_0, a], g \in G_a, \delta = [a, b]$. Then

$$\begin{aligned}\mathrm{Stab}(\delta) &= \mathrm{Stab}(v) \cap \mathrm{Stab}(vge) = vG_a v^{-1} \cap (vge)G_b(vge)^{-1} \\ &= vg(G_a \cap eG_b e^{-1})g^{-1}v^{-1} = vg(\alpha_{\bar{e}}(G_e))g^{-1}v^{-1}\end{aligned}$$

As before, the equality above is in $F(G, Y)$.

The subgroup $\alpha_{\bar{e}}(G_e)$ of G_a appears as a subgroup $\hat{G}_e$ of H via the map $g \mapsto \hat{g} = g_a g g_a^{-1}$. Thus

$$\mathrm{Stab}(\delta) = vgg_a^{-1}\hat{G}_e g_a g^{-1} v^{-1} = h\hat{G}_e h^{-1}, \text{ with } h = vgg_a^{-1} \in H.$$

We denote the tree thus obtained $\mathcal{T}(G, Y, a_0)$ and we call it the universal covering tree or the Bass–Serre tree of the graph of groups (G, Y). □

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