Geometric Group Theory

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Chapter 1

Introduction

Geometric group theory is a descendant of combinatorial group theory, which in turn is the study of groups using their presentations. So one studies mainly infinite, finitely generated groups and is more interested in the class of finitely presented groups. Combinatorial group theory was developed in close connection to low dimensional topology and geometry.

The fundamental group of a compact manifold is finitely presented. So finitely presented groups give us an important invariant that helps us distinguish manifolds. Conversely topological techniques are often useful for studying groups. Dehn in 1912 posed some fundamental algorithmic problems for groups: The word problem, the conjugacy problem and the isomorphism problem. He solved these problems for fundamental groups of surfaces using hyperbolic geometry. Later the work of Dehn was generalized by Magnus and others, using combinatorial methods.

In recent years, due to the fundamental work of Stallings, Serre, Rips, Gromov powerful geometric techniques were introduced to the subject and combinatorial group theory developed closer ties with geometry and 3-manifold theory. This led to important results in 3-manifold theory and logic.

Some leitmotivs of combinatorial/geometric group theory are:

1. Solution of the fundamental questions of Dehn for larger classes of groups. One should remark that Novikov and Boone in the 50's showed that Dehn's problems are unsolvable in general. One may think of finitely presented groups as a jungle. The success of the theory is that it can deal with many natural classes of groups which are also important for topology/geometry. As we said the first attempts at this were combinatorial in nature, one imposed the so-called small cancelation conditions on the presentation. This was subsequently geometrized using van-Kampen diagrams by Lyndon-Schupp.

Gromov in 1987 used ideas coming from hyperbolic geometry to show that algorithmic problems can be solved for a very large ('generic') class of groups (called hyperbolic groups). It was Gromov's work that demonstrated that the geometric point of view was very fruitful for the study of groups and created geometric group theory. We will give a brief introduction to the theory of hyperbolic groups in the last sections of these notes.

2. One studies the structure of groups, in particular the subgroup structure. Ideally one would want to describe all subgroups of a given group. Some particular questions of interest are: existence of subgroups of finite index, existence of normal subgroups, existence of free subgroups and of free abelian subgroups etc.

Another structural question is the question of the decomposition of a group in 'simpler' groups. One would like to know if a group is a direct product, free product, amalgamated product etc. Further one would like to know if there is a canonical way to decompose a group in these types of products. The simplest example of such a theorem in the decomposition of a finitely generated abelian group as a direct product of cyclic groups.

In this course we will focus on an important tool of geometric group theory: the study of groups via their actions on trees, this is related to both structure theory and the subgroup structure of groups.

3. Construction of interesting examples of groups. Using amalgams and HNN extensions Novikov and Boone constructed finitely presented groups with unsolvable word and conjugacy problem. We mention also the Burnside question: Are there infinite finitely generated torsion groups? What about torsion groups of bounded exponent? The answer to both of these is yes (Novikov) but to this date it is not known whether there are infinite, finitely presented torsion groups.

Some of the recent notable successes of the theory is the solution of the Tarski problem by Sela and the solution of the virtually Haken conjecture and the virtually fibering conjecture by Agol-Wise.

The Tarski problem was an important problem in Logic asking whether the free groups of rank 2 and 3 have the same elementary theory i.e. whether the set of sentences which are valid in F_2 is the same with the set of sentences valid in F_3. Somewhat surprisingly the positive solution to this uses actions on Trees and Topology (and comprises more than 500 pages!).

The solution of the virtually Haken conjecture and the virtually fibering conjecture by Agol-Wise implies that every closed 3-manifold can be 'build' by gluing manifolds that are quite well understood topologically and after the fundamental work of Perelman completed our picture of what 3-manifolds look like. More explicitly an obvious way to construct 3-manifolds is by taking a product of a surface with $[0, 1]$ and then gluing the two boundary surface pieces by a homeomorphism. The result of Agol-Wise shows that every 3-manifold can be build from pieces that have a finite sheeted cover that is either S^3 or of the form described in the previous sentence.

Chapter 2

Free Groups

Definition 2.1. Let X be a subset of a group F. We say that F is a *free group with basis* X if any function φ from X to a group G can be extended uniquely to a homomorphism $\bar{\varphi} : F \to G$.

One may remark that the trivial group $\{e\}$ is a free group with basis the empty set. Also the infinite cyclic group $C =< a >$ is free with basis $\{a\}$. Indeed if G is any group and if $\varphi(a) = g$ then φ is extended to a homomorphism by

$$\bar{\varphi}(a^n) = \varphi(a)^n, \ \ n \in \mathbb{Z}$$

It is clear that this extension is unique. So $\{a\}$ is a free basis of C. We remark that $\{a^{-1}\}$ is another free basis of C.

Proposition 2.1. *Let X be a set. Then there is a free group $F(X)$ with basis X.*

Proof. We consider the set $S = X \sqcup X^{-1}$ where $X^{-1} = \{s^{-1} : s \in X\}$. A word on X is a finite sequence $(s_1, ..., s_n)$ where $s_i \in S$. We denote by e the empty sequence. We usually denote words as strings of letters, so eg if (a, a^{-1}, b, b) is a word we write simply $aa^{-1}bb$ or $aa^{-1}b^2$. Let W be the set of words on S. We define an equivalence relation $\sim$ on W generated by:

$$uaa^{-1}v \sim uv, \ ua^{-1}av \sim uv \quad \text{for any } a \in S, \ u, v \in W$$

So two words are equivalent if we can go to one from the other by a finite sequence of insertions and/or deletions of consecutive inverse letters.

Let $F := W/\sim$ be the set of equivalence classes of this relation. We denote by $[w]$ the equivalence class of $w \in W$. If

$$w = (a_1, ..., a_n), \ v = (b_1, ..., b_k)$$

then we define the product wv of w, v by

$$wv = (a_1, ..., a_n, b_1, ..., b_k)$$

We remark that if $w_1 \sim w_2$, $v_1 \sim v_2$ then $w_1 v_1 \sim w_2 v_2$, so we define multiplication on F by $[w][v] = [wv]$. We claim that F with this operation is a group. Indeed $e = [\emptyset]$ is the identity element and if $w = (b_1, ..., b_n)$ the inverse element is given by $w^{-1} = (b_n^{-1}, ..., b_1^{-1})$. Here we follow the usual convention that if $s^{-1} \in X^{-1}$ then $(s^{-1})^{-1} = s$. It is clear that associativity holds:

$$([w][u])[v] = [w]([u][v])$$

since both sides are equal to $[wuv]$.

If $w \in W$ we denote by $|w|$ the length of w (eg $|aa^{-1}ba| = 4$). We say that a word w is *reduced* if it does not contain a subword of the form aa^{-1} or $a^{-1}a$ where $a \in X$. To complete the proof of the theorem we need the following:

Lemma 2.1. *Every equivalence class $[w] \in F$ has a unique representative which is a reduced word.*

Proof. It is clear that $[w]$ contains a reduced word. Indeed one starts with w and eliminates successively pairs of the form aa^{-1} or $a^{-1}a$ till none are left. What this lemma says is that the order under which eliminations are performed doesn't matter. This is quite obvious but we give here a formal (and rather tedious) argument.

It is enough to show that two distinct reduced words w, v are not equivalent. We argue by contradiction. If w, v are equivalent then there is a sequence

$$w_0 = w, w_1, ..., w_n = v$$

where each w_{i+1} is obtained from w_i by insertion or deletion of a pair of the form aa^{-1} or $a^{-1}a$. We assume that for the sequence w_i the sum of the lengths $L = \sum |w_i|$ is the minimal possible among all sequences of this type going from w to v. Since w, v are reduced we have that $|w_1| > |w_0|$, $|w_{n-1}| > |w_n|$. It follows that for some i we have

$$|w_i| > |w_{i-1}|, \ |w_i| > |w_{i+1}|$$

So w_{i-1} is obtained from w_i by deletion of a pair a, a^{-1} and w_{i+1} is obtained from w_i by deletion of a pair b, b^{-1}. If these two pairs are distinct in w_i then we can delete b, b^{-1} first and then add a, a^{-1} decreasing L. More precisely if we have for instance

$$w_i = u_1 bb^{-1} u_2 aa^{-1} u_3, \ w_{i-1} = u_1 bb^{-1} u_2 u_3, \ w_{i+1} = u_1 u_2 aa^{-1} u_3$$

we can replace w_i by $u_1 u_2 u_3$. In this way L decreases by 4, which is a contradiction.

Now if the pairs a, a^{-1}, b, b^{-1} are not distinct we remark that $w_{i-1} = w_{i+1}$ which is again a contradiction. $\square$

We can now identify X with the subset $\{[s] : s \in X\}$ of F. Let G be a group and let $\varphi : X \to G$ be any function. Then we define a homomorphism $\bar{\varphi} : F \to G$ as follows: if $s^{-1} \in X^{-1}$ we define $\bar{\varphi}(s^{-1}) = \varphi(s)^{-1}$. If $w = s_1 ... s_n$ is a reduced word we define

$$\bar{\varphi}([w]) = \varphi(s_1)...\varphi(s_n)$$

It is easy to see that $\bar{\varphi}$ is a homomorphism. We remark finally that this extension of φ is unique by definition. So $F(X) = F$ is a free group with basis X. $\square$

Using the lemma above we can identify the elements of F with the reduced words of W.

Remark 2.1. In the sequel if w is any word in X (not necessarily reduced) we will also consider w as an element of the free group $F(X)$. This could cause some confusion as it is possible to have $w \neq v$ as words but $w = v$ in $F(X)$.

Proposition 2.2. *Every map $\varphi : X \to H$, where H is a group, has a unique extension to a group homomorphism $\Phi : F(X) \to H$.*

Proof. The map $\varphi : X \to H$ has an obvious extension $\bar{\varphi} : X \sqcup X^{-1} \to H$. The homomorphisms Φ is defined by $\Phi(w_\emptyset) = 1_H$ and $\Phi(a_1 \dots a_n) = \bar{\varphi}(a_1) \cdots \bar{\varphi}(a_n)$ for every word $a_1 \dots a_n$ in the alphabet $X \sqcup X^{-1}$. The uniqueness follows from the fact that every group homomorphism is uniquely determined by its restriction on a generating set. $\square$

Corollary 2.1. *Every group is a quotient group of a free group.*

Proof. Let G be a group. We consider the free group with basis G, $F(G)$. If $\varphi : G \to G$ is the identity map $\varphi(g) = g$, then φ can be extended to an epimorphism $\bar{\varphi} : F(G) \to G$. If $N = \mathit{ker}\,(\bar{\varphi})$ then

$$G \cong F(G)/N$$

$\square$

If X is a set we denote by $|X|$ the cardinality of X.

Corollary 2.2. *Consider two groups G and H, G generated by a subset X.*

1. *The map* $\mathrm{Hom}(G, H) \to \mathrm{Map}(X, H)$ *defined by restricting group homomorphisms to X is injective. In particular* $|\mathrm{Hom}(G, H)| \leq |H|^{|X|}$.

2. *If moreover $G = F(X)$ then the map defined in* (1) *is a bijection and* $|\mathrm{Hom}(G, H)| = |H|^{|X|}$.

Proof. (1) follows from the fact already used above, that every group homomorphism is uniquely determined by its restriction on a generating set.

(2) The fact that the map defined in (1) is onto is a consequence of Proposition 2.2. □

Proposition 2.3. *Let $F(X), F(Y)$ be free groups on X, Y. Then $F(X)$ is isomorphic to $F(Y)$ if and only if $|X| = |Y|$.*

Proof. Assume that $|X| = |Y|$. We consider a 1-1 and onto function $f : X \to Y$. Let $h = f^{-1}$. The maps f, h are extended to homomorphisms $\bar{f}, \bar{h}$ and $\bar{f} \circ \bar{h}$ is the identity on $F(Y)$ while $\bar{h} \circ \bar{f}$ is the identity on $F(X)$ so $\bar{f}$ is an isomorphism.

Conversely assume that $F(X)$ is isomorphic to $F(Y)$. If X, Y are infinite sets then the cardinality of $F(X)$, $F(Y)$ is equal to the cardinality, respectively of X, Y. So if these groups are isomorphic $|X| = |Y|$. Otherwise if, say, $|X|$ is finite, we note that there are $2^{|X|}$ homomorphisms from $F(X)$ to $\mathbb{Z}_2$. Since $F(X) \cong F(Y)$ we have that $2^{|X|} = 2^{|Y|}$ so $|X| = |Y|$. □

Corollary 2.3. *Let $F(X)$ be a free group on X. If A is any set of generators of $F(X)$ then $|A| \geq |X|$.*

Proof. Indeed if $|A| < |X|$ then there are at most $2^{|A|}$ homomorphisms from $F(X)$ to $\mathbb{Z}_2$, a contradiction. □

If F is a free group with free basis X then the *rank* of F is the cardinality of X. We denote by F_n the free group of rank n.

The word problem

If F is a free group with free basis X then we identify the elements of F with the words in X. This is a bit ambiguous as equivalent words represent the same element. The word problem in this case is to decide whether a word represents the identity element. This is of course trivial as it amounts to checking whether the word reduces to the empty word after cancelations.

The conjugacy problem

Definition 2.2. If $w = s_1 ... s_n$ is a word then the cyclic permutations of w are the words:

$$s_n s_1 ... s_{n-1}, \; s_{n-1} s_n ... s_{n-2}, \; ....., \; s_2 ... s_n s_1$$

A word is called *cyclically reduced* if it is reduced and all its cyclic permutations are reduced words.

We remark that a word w on S is cyclically reduced if w is reduced and $w \neq xvx^{-1}$ for any $x \in S \sqcup S^{-1}$.

Proposition 2.4. *Let $F(X)$ be a free group. Every word $w \in F(X)$ is conjugate to a cyclically reduced word. Two cyclically reduced words w, v are conjugate if and only if they are cyclic permutations of each other.*

Proof. Let r be a word of minimal length that is conjugate to w. If $r = xux^{-1}$ then r is conjugate to u and $|u| < |r|$ which is a contradiction. Hence r is cyclically reduced.

Let w now be a cyclically reduced word. Clearly every cyclic permutation of w is conjugate to w. We show that a cyclically reduced word conjugate to w is a cyclic permutation of w. We argue by contradiction.

Let g be a word of minimal length such that the reduced word v representing $g^{-1}wg$ is cyclically reduced but is not a cyclic permutation of w. If the word gvg^{-1} is reduced then it is not cyclically reduced. But $w = gvg^{-1}$ and w is cyclically reduced so gvg^{-1} is not reduced. If $g = s_1...s_n$, $s_i \in X \cup X^{-1}$ then either $v = s_n^{-1}u$ or $v = us_n$. If $v = s_n^{-1}u$ then

$$gvg^{-1} = s_1...s_{n-1}(us_n^{-1})(s_1...s_{n-1})^{-1}$$

By our assumption that g is minimal length we have that us_n^{-1} is a cyclic permutation of w. But then $v = s_n^{-1}u$ is also a cyclic permutation of w. We argue similarly if $v = us_n$. □

Using this proposition it is easy to solve algorithmically the conjugacy problem in a free group.

Remark 2.2. A word g is cyclically reduced if and only if gg is reduced. Clearly if a word w is reduced then $w = uvu^{-1}$ where v is cyclically reduced.

Proposition 2.5. *A free group F has no elements of finite order.*

Proof. Let $g \in F$. Then g is conjugate to a cyclically reduced word h. Clearly g, h have the same order. We remark now that h^n is reduced for any $n \in \mathbb{N}$ so $h^n \neq e$, ie the order of g is infinite. □

Proposition 2.6. *Let F be a free group and $g, h \in F$. If $g^k = h^k$ for some $k \geq 1$ then $g = h$.*

Proof. Let's say that $g = ug_1u^{-1}$ with $u \in F$ and g_1 cyclically reduced. Then $g_1^k = (u^{-1}gu)^k = (u^{-1}hu)^k$. Let h_1 be the reduced word equal to $u^{-1}hu$.

If h_1 is not cyclically reduced then $g_1^k \neq h_1^k$ since h_1^k is not cyclically reduced. Otherwise

$$g_1^k = h_1^k \implies g_1 = h_1$$

since g_1^k, h_1^k are reduced words. Hence $g = h$. □

Exercises 2.1. 1. Show that F_2 has a free subgroup of rank 3.

2. Show that F_2 has a free subgroup of infinite rank.

Chapter 3

Finitely generated and finitely presented groups

3.1 Finitely generated groups

A group that has a finite generating set is called *finitely generated*.

Definition 3.1. The *rank* of a finitely generated group G, denoted $\text{rank}\,(G)$, is the minimal number of generators of G.

Corollary 2.3 implies that the rank of a free group $F(X)$ is $|X|$.

Exercise 3.2. Show that every finitely generated group is countable.

Remark 3.1. While finitely generated groups are comparatively small (since countable) their class is large: the number of isomorphism classes of finitely generated groups has power continuum.

Examples 3.3.

1. The group $(\mathbb{Z}, +)$ is finitely generated by both $\{1\}$ and $\{-1\}$. Also, any set $\{p, q\}$ of coprime integers generates $\mathbb{Z}$.
2. The group $(\mathbb{Q}, +)$ is not finitely generated.
3. The transposition (12) and the cycle $(12 \dots n)$ generate the permutation group S_n.

Proposition 3.1.

1. *Every quotient $\bar{G}$ of a finitely generated group G is finitely generated; we can take as generators of $\bar{G}$ the images of the generators of G.*

2. *The converse of the above statement is clearly not true, the fact that a quotient G/N is finitely generated does not imply anything on the group, one needs to add an extra assumption: if N is a normal subgroup of G, and both N and G/N are finitely generated, then G is finitely generated.*

3. *A subgroup of a finitely generated group is not necessarily finitely generated (see Ex. 2, Problem Sheet 1). Not even if the subgroup is normal.*

4. *A finite index subgroup of a finitely generated group is finitely generated.*

3.2 Presentations

Definition 3.4. A *presentation* P is a pair $P = \langle S|R \rangle$ where S is a set and R is a set of words in S. The group defined by P is the quotient group

$$G = F(S)/\langle\langle R \rangle\rangle$$

where $\langle\langle R \rangle\rangle$ is the smallest normal subgroup of the free group $F(S)$ that contains R. By abuse of notation we write often $G = \langle S|R \rangle$.

Remark 3.2. From corollary 2.1 it follows that every group has a presentation.

A group G is called *finitely generated* if there are finitely many elements of G, $g_1, ..., g_n$ such that any element of g can be written as a product of $g_i^{\pm 1}$, $i = 1, ..., n$. Clearly if G is finitely generated then G has a presentation $\langle S|R \rangle$ with S finite. We say that a group $G = \langle S|R \rangle$ is *finitely related* if R is finite. If both S and R are finite we say that G is *finitely presented*. S is the set of generators and R is the set of relators of the presentation. Sometimes we write relators as equations, so instead of writing r we write $r = 1$ or even $r_1 = r_2$, which is of course equivalent to $r_1 r_2^{-1} = 1$.

Examples. 1. A presentation of $\mathbb{Z}$ is given by $\langle a| \ \rangle$.

2. A presentation of $\mathbb{Z}_n$ is given by $\langle a|a^n \rangle$.

3. A presentation of the free group $F(S)$ is given by $\langle S| \ \rangle$.

4. A presentation of $\mathbb{Z} \oplus \mathbb{Z}$ is given by $\langle a, b|aba^{-1}b^{-1} \rangle$.

Indeed if $\varphi : F(a, b) \to \mathbb{Z} \oplus \mathbb{Z}$ is the homomorphism defined by $\varphi(a) = (1, 0)$, $\varphi(b) = (0, 1)$ then clearly $aba^{-1}b^{-1} \in ker\, \varphi$. We set $N = \langle\langle aba^{-1}b^{-1} \rangle\rangle$. Since $aba^{-1}b^{-1} \in ker\, \varphi$, $N \subset ker\, \varphi$. We remark that in $F(a, b)/N$ we have that $ab = ba$. If

$$w = a^{k_1} b^{m_1} ... a^{k_n} b^{m_n} \in ker\, \varphi$$

then $\sum k_i = \sum m_i = 0$. Therefore $w = 1$ in $F(a,b)/N$ since $ab = ba$ in this quotient group. It follows that $ker\,\varphi \subset N$ and $\langle a, b | aba^{-1}b^{-1}\rangle$ is a presentation of $\mathbb{Z} \oplus \mathbb{Z}$.

5. If G is a finite group, $G = \{g_1, ..., g_n\}$ then a presentation of G is: $\langle G|R\rangle$ where R is the set of the n^2 equations of the form $g_i g_j = g_k$ given by the multiplication table of G.

6. The presentation $\langle a, b | a^{-1}ba = b^2, b^{-1}ab = a^2\rangle$ is a presentation of the trivial group. Indeed

$$a^{-1}ba = b^2 \implies (b^{-1}a^{-1}b)a = b \implies a^{-1} = b \implies a = 1 = b$$

Remark 3.3. Let $G = \langle S|R\rangle$. Then a word w on S represents the identity in G if and only if w lies in the normal closure of R in $F(S)$. Equivalently if w can be written in $F(S)$ as a product of conjugates of elements of R :

$$w = \prod_{i=1}^{n} x_i r_i^{\pm 1} x_i^{-1}, \;\; r_i \in R,\, x_i \in F(S)$$

We note that if w represents the identity in G we could prove that it is the case by listing all expressions of this form. Eventually we will find one such expression that is equal to w in S. Of course this presupposes that we know that $w = 1$ in G, otherwise this process will never terminate.

Proposition 3.2. *Let $G = \langle S|R\rangle$ and let H be a group. If $\varphi : S \to H$ is a function then φ can be extended to a homomorphism $\bar{\varphi} : G \to H$ if and only if $\varphi(r) = 1$ for every $r \in R$, where if r is the word $a_1^{\pm 1}...a_n^{\pm 1}$, we define $\varphi(r) = \varphi(a_1)^{\pm 1}...\varphi(a_n)^{\pm 1}$.*

Proof. It is obvious that $\varphi(r) = 1$ for every $r \in R$ is a necessary condition for φ to extend to a homomorphism.

Clearly φ extends to $\varphi : F(S) \to H$. Assume now that $\varphi(r) = 1$ for every $r \in R$. If $N = \langle\langle R\rangle\rangle$ then clearly $N \subset ker\,\varphi$. So the map $\bar{\varphi}(aN) = \varphi(a)$ is a well defined homomorphism from $G = F(S)/N$ to H that extends φ.

□

One can use this proposition to show that a group given by a presentation is non trivial by finding a non trivial homomorphism to another group.

Before the next example we recall the definition of the semidirect product:

Let A, B be groups and let $\varphi : B \to Aut(A)$ be a homomorphism. Then we define the semidirect product of A and B to be the group $G = A \rtimes_\varphi B$ with elements the elements of the Cartesian product $A \times B$ and operation defined by

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1\varphi(b_1)(a_2), b_1 b_2)\ .$$

Example 3.1. If $G = \langle a, t | tat^{-1} = a^2 \rangle$ then $< t > \cong < a > \cong \mathbb{Z}$.

Proof. Consider the subgroup of $\mathbb{Q}$:

$$\mathbb{Z}\left[\frac{1}{2}\right] = \{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$$

We define an isomorphism $\varphi : \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}]$, by $\varphi(x) = 2x$. We form now the semidirect product $\mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{Z}[\frac{1}{2}]$ via φ. The elements of this semidirect product can be written as pairs $(\frac{m}{2^n}, k)$. We define now

$$\psi : G \to \mathbb{Z}\left[\frac{1}{2}\right] \rtimes \mathbb{Z}, \quad \text{by } \psi(a) = (1,0), \ \psi(t) = (0,1).$$

Since

$$\psi(tat^{-1}) = \psi(a^2) = (2,0)\,,$$

ψ is a homomorphism. Since a, t map to infinite order elements we have that $< t > \cong < a > \cong \mathbb{Z}$.

□

3.3 Finitely presented groups

From the viewpoint of the algorithmic approach towards the understanding of infinite groups, finitely presented groups are the most convenient, because describable by finite data.

Remark 3.4. The family of finitely presented groups is countable.

It is important to understand whether being finitely presented is a feature of the group, or a feature of a pair (group, generating set). The following result shows that it is indeed a feature of the group.

Remark 3.5. Let $G = \langle X \rangle = \langle Y \rangle$ and X is finite then there is a finite subset $Y' \subset Y$ such that $G = \langle Y' \rangle$.

Thus, to begin with, finite generation does not depend on the generating set we pick. The next proposition shows that something similar holds for finite presentability.

Proposition 3.3. *Let $G \cong \langle S | R \rangle \cong \langle X | Q \rangle$ where S, X, R are finite. Then there is a finite subset Q' of Q such that $G \cong \langle X | Q' \rangle$*

Proof. Let $\varphi : F(S)/\langle\langle R\rangle\rangle \to F(X)/\langle\langle Q\rangle\rangle$ be an isomorphism. Let

$$S = \{s_1, ..., s_n\},\ R = \{r_1, ..., r_k\},\ X = \{x_1, ..., x_m\}$$

Then the r_i's are words in the s_j's, $r_i = r_i(s_1, ..., s_n)$. Let $\varphi(s_i) = s_i'$, $i = 1, ..., n$. If we see the s_i' as elements of $F(X)$, since φ is onto we have that the generators of G can be written in terms of the s_i', so there are words $w_j(s_1', ..., s_n')$, $j = 1, ..., m$ and $u_1, ..., u_m \in \langle\langle Q\rangle\rangle$ such that

$$x_j = w_j(s_1', ..., s_n')u_j,\ j = 1, ..., m$$

where the equality is in $F(X)$. Since φ is a homomorphism we have also that

$$r_i(s_1', ..., s_n') = v_i \in \langle\langle Q\rangle\rangle,\ i = 1, ..., k$$

Let Q' be a finite subset of Q such that all u_j, v_i, $j = 1, ..., m$, $i = 1, ..., k$ can be written as products of conjugates of elements of Q'. We claim that $\langle\langle Q'\rangle\rangle = \langle\langle Q\rangle\rangle$. Indeed the map

$$\psi : F(S)/\langle\langle R\rangle\rangle \to F(X)/\langle\langle Q'\rangle\rangle$$

given by $\psi(s_i) = s_i'$ is an onto homomorphism and $\varphi = \pi \circ \psi$ where π is the natural quotient map

$$\pi : F(X)/\langle\langle Q'\rangle\rangle \to F(X)/\langle\langle Q\rangle\rangle$$

However φ is 1-1 so π is also 1-1. It follows that $\langle\langle Q'\rangle\rangle = \langle\langle Q\rangle\rangle$. □

The next important step is to understand what, if anything, connects two finite presentations of the same group.

3.4 Tietze transformations

Different presentations of the same group are related via Tietze transformations. There are two types of Tietze transformations:

(T1) If $\langle S|R\rangle$ is a presentation and $r \in \langle\langle R\rangle\rangle \subset F(S)$ then T1 is the replacement of $\langle S|R\rangle$ by $\langle S|R \cup \{r\}\rangle$. Clearly these two presentations define isomorphic groups, an isomorphism ϕ is defined on the generators by $\phi(s) = s$ for all $s \in S$.

We denote also by T1 the inverse transformation.

(T2) If $\langle S|R\rangle$ is a presentation, $a \notin S$ and $w \in F(S)$ then T2 is the replacement of $\langle S|R\rangle$ by $\langle S \cup \{a\}|R \cup \{a^{-1}w\}\rangle$. Clearly these two presentation define isomorphic groups. A homomorphism ϕ is defined on the generators by $\phi(s) = s$ for all $s \in S$. One verifies easily that the inverse of ϕ is given by $\psi(s) = s$ for all $s \in S$ and $\psi(a) = w$.

We denote also by T2 the inverse transformation.

Theorem 3.1. *Two finite presentations $\langle S_1|R_1\rangle$, $\langle S_2|R_2\rangle$ define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.*

Proof. It is clear that if two presentations are related by a finite sequence of Tietze transformations they define isomorphic groups. Conversely suppose that $G_1 = \langle S_1|R_1\rangle \cong \langle S_2|R_2\rangle = G_2$. We may assume that $S_1 \cap S_2 = \emptyset$. Indeed if this is not the case using moves T1, T2 we can replace S_1 by a set of letters with the same cardinality, disjoint from S_2. We consider now isomomorphisms

$$\varphi : G_1 \to G_2, \ \ \psi = \varphi^{-1} : G_2 \to G_1$$

For each $s \in S_1, t \in S_2$ consider words w_s, v_t such that $\varphi(s) = w_s, \ \psi(t) = v_t$. Let

$$U_1 = \{s^{-1}w_s : s \in S_1\}, \ \ U_2 = \{t^{-1}v_t : t \in S_2\}$$

We consider the presentation:

$$\langle S_1 \cup S_2|R_1 \cup R_2 \cup U_1 \cup U_2\rangle$$

We claim that there is a finite sequence of Tietze transformations from $\langle S_1|R_1\rangle$ to this presentation. Indeed using $T2$ we may introduce one by one the generators of S_2 and the relations U_2. So we obtain the presentation

$$\langle S_1 \cup S_2|R_1 \cup U_2\rangle$$

The Tietze transformations give as an isomorphism

$$\rho : \langle S_1 \cup S_2|R_1 \cup U_2\rangle \to \langle S_1|R_1\rangle$$

where $\rho(s) = s$, $\rho(t) = v_t$ for $s \in S_1$, $t \in S_2$. We remark that $\varphi \circ \rho$ is a homomorphism from $\langle S_1 \cup S_2|R_1 \cup U_2\rangle$ to $\langle S_2|R_2\rangle$ and $\varphi \circ \rho(t) = t$ for all $t \in S_2$. It follows that for any $r \in R_2$, $\varphi \circ \rho(r) = r = 1$, hence $R_2 \subseteq \langle\langle R_1 \cup U_2\rangle\rangle$. So using $T1$ we obtain the presentation

$$\langle S_1 \cup S_2|R_1 \cup R_2 \cup U_2\rangle$$

We remark now that $\varphi \circ \rho$ is still defined on this presentation and $\varphi \circ \rho(s) = w_s$ for all $s \in S_1$, while $\varphi \circ \rho(w_s) = w_s$. It follows that $s^{-1}w_s$ is mapped to 1 by $\varphi \circ \rho$, hence the relators U_1 also follow from $R_1 \cup R_2 \cup U_2$. So applying T1 we obtain

$$\langle S_1 \cup S_2|R_1 \cup R_2 \cup U_1 \cup U_2\rangle$$

Similarly we see that there is a finite sequence of Tietze transformations from $\langle S_2|R_2\rangle$ to this presentation. $\square$

Proposition 3.4. *Suppose that N a normal subgroup of a group G. If both N and G/N are finitely presented then G is also finitely presented.*

Proof. Let X be a finite generating set of N and let Y be a finite subset of G such that $\overline{Y} = \{yN \mid y \in Y\}$ is a generating set of G/N. Let $\langle X \mid r_1, \ldots, r_k\rangle$ be a finite presentation of N and let $\langle \overline{Y} \mid \rho_1, \ldots, \rho_m\rangle$ be a finite presentation of G/N. The group G is generated by $S = X \cup Y$ and this set of generators satisfies a list of relations of the following form:

$$r_i(X) = 1\,,\ 1 \leqslant i \leqslant k\,,\ \rho_j(Y) = u_j(X)\,,\ 1 \leqslant j \leqslant m\,, \tag{3.1}$$

$$x^y = v_{xy}(X)\,,\ x^{y^{-1}} = w_{xy}(X) \tag{3.2}$$

for some words u_j, v_{xy}, w_{xy} in S.

We claim that this is a complete set of defining relators of G.

All the relations above can be rewritten as $t(X,Y) = 1$ for a finite set T of words t in S. Let K be the normal subgroup of $F(S)$ normally generated by T.

The epimorphism $\pi_S : F(S) \to G$ defines an epimorphism $\varphi : F(S)/K \to G$. Let wK be an element in $\ker(\varphi)$, where w is a word in S. Due to the set of relations (3.2), there exist a word $w_1(X)$ in X and a word $w_2(Y)$ in Y, such that $wK = w_1(X)w_2(Y)K$.

Applying the projection $\pi : G \to G/N$, we see that $\pi(\varphi(wK)) = 1$, i.e. $\pi(\varphi(w_2(Y)K)) = 1$. This implies that $w_2(Y)$ is a product of finitely many conjugates of $\rho_i(Y)$, hence $w_2(Y)K$ is a product of finitely many conjugates of $u_j(X)K$, by the second set of relations in (3.1). This and the relations (3.2) imply that $w_1(X)w_2(Y)K = v(X)K$ for some word $v(X)$ in X. Then the image $\varphi(wK) = \varphi(v(X)K)$ is in N; therefore, $v(X)$ is a product of finitely many conjugates of relators $r_i(X)$. This implies that $v(X)K = K$.

We have thus obtained that $\ker(\varphi)$ is trivial, hence φ is an isomorphism, equivalently that $K = \ker(\pi_S)$. This implies that $\ker(\pi_S)$ is normally generated by the finite set of relators listed in (3.1) and (3.2). $\square$

Proposition 3.5. *Let G be a group, and $H \leq G$ such that $|G : H|$ is finite. Then G is FP if and only if H is FP.*

Proof. Suppose that $G = \langle X; R\rangle$ with X and R finite. We have an epimorphism $\pi : F = F(X) \to G$ with $K = \ker \pi = \langle\langle R\rangle\rangle$. Put $E = \pi^{-1}(H)$. Then $|F : E| = |G : H|$ is finite, so E is free on some finite basis Y. Since $K \leq E$, each $r \in R$ satisfies $r = s_r(Y)$

for some word s_r on Y. Put $S = \{s_r(Y) \mid r \in R\}$. Then $\pi_1 = \pi_{|E} : E \to H$ is an epimorphism and

$$\ker \pi_1 = K = \left\langle \left\langle S^F \right\rangle \right\rangle = \left\langle S^F \right\rangle .$$

Say $F = a_1 E \cup \ldots \cup a_n E$. Then $S^F = (S^{a_1} \cup \ldots \cup S^{a_n})^E$. Thus $\langle Y; S^{a_1} \cup \ldots \cup S^{a_n} \rangle$ is a presentation for H.

Suppose conversely that H is FP. Let $N \leq H$ be a normal subgroup of finite index in G. Then $|H : N|$ is finite, so N is FP by the first part. Also G/N is FP. Therefore G is FP by Proposition 3.4. □

3.5 Algorithmic problems in group theory/Dehn's problems

Presentations provide a 'compact' form of defining a group. They were introduced by Max Dehn in the early 20-th century. A central topic in combinatorial group theory is to derive algebraic information about a group from its presentation. Below is a list of problems formulated in this spirit, whose origin lies in the work of Max Dehn in the early 20th century.

Word Problem. Let $G = \langle S|R \rangle$ be a finitely presented group. Design an algorithm/construct a Turing machine (or prove its non-existence) that, given a word w in the generating set X as its input, would determine if w represents the trivial element of G, i.e. if

$$w \in \langle\langle R \rangle\rangle.$$

Conjugacy Problem. Let $G = \langle S|R \rangle$ be a finitely presented group. Design an algorithm/construct a Turing machine (or prove its non-existence) that, given a pair of word v, w in the generating set X, would determine if v and w represent conjugate elements of G, i.e. if there exists $g \in G$ so that

$$[w] = g^{-1}[v]g.$$

To simplify the language, we will state such problems below as: Given a finite presentation of G, determine if two elements of G are conjugate.

Isomorphism Problem. Given two (finite) presentations $G_i = \langle X_i | R_i \rangle, i = 1, 2$ as an input, determine if G_1 is isomorphic to G_2.

A particular case of the Isomorphism Problem is the following.

Triviality Problem. Given a (finite) presentation $G = \langle S|R \rangle$ as an input, determine if G is trivial, i.e. equals $\{1\}$.

Other well known algorithmic problems are the following.

Embedding Problem. Given two (finite) presentations $G_i = \langle X_i | R_i \rangle, i = 1, 2$ as an input, determine if G_1 is isomorphic to a subgroup of G_2.

Membership Problem. Let G be a finitely presented group, $h_1, \dots, h_k \in G$ and H, the subgroup of G generated by the elements h_i. Given an element $g \in G$, determine if g belongs to H.

Note that a group with solvable conjugacy or membership problem, also has solvable word problem.

It was discovered in the 1950-s in the work of P. S. Novikov, W. Boone and M. O. Rabin that all of the above problems are *algorithmically unsolvable*. For instance, in the case of the word problem, given a finite presentation $G = \langle S | R \rangle$, there is no algorithm whose input would be a (reduced) word w and the output YES is $w \equiv_G 1$ and NO if not. A. A. Fridman (1960) proved that certain groups have solvable word problem and unsolvable conjugacy problem. There are also examples of groups with solvable word and conjugacy problems but unsolvable membership problem.

Nevertheless, the main message of the Geometric Group Theory is that *under various geometric assumptions on groups (and their subgroups), all of the above algorithmic problems are solvable.* Incidentally, the idea that geometry can help solving algorithmic problems also goes back to Max Dehn.

Proposition 3.6. *If the word problem is solvable for the finite presentation $\langle S|R \rangle$ of a group G then it is solvable for any other finite presentation $\langle X|Q \rangle$ of G. The same is true for the conjugacy problem.*

Proof. To solve the word problem, given a word w on X we run 'in parallel' two procedures:

(1) We list all elements in $\langle\langle Q \rangle\rangle$ in $F(X)$ and we check whether w is equal to one of these words in $F(X)$.

In other words, this procedure multiplies conjugates of the relators $q_j \in Q$ by products of the generators $x_i \in X$ (and their inverses), and transforms the product into a reduced word. Every element of $\langle\langle Q \rangle\rangle$ is such a product, of course.

If we find that w equals one of the elements of $\langle\langle Q \rangle\rangle$, then we stop and conclude that $w \equiv_G 1$.

(2) We list all homomorphisms $\varphi : F(X)/\langle\langle Q \rangle\rangle \to F(S)/\langle\langle R \rangle\rangle$. To find φ we enumerate $|X|$-tuples of words in $F(S)$ and we check for each such choice whether the relations Q are satisfied. We note that this is possible to do since the word problem is solvable in

$\langle S|R\rangle$. Given a homomorphism φ we check whether $\varphi(w) \neq 1$ (which is likewise possible to do since the word problem is solvable in $\langle S|R\rangle$).

Clearly one of the procedures (1),(2) will terminate.

To solve the conjugacy problem, given two words w, v on X we argue similarly.

We run the following procedures 'in parallel':

(1.a) We list all elements of the form $gvg^{-1}w^{-1}$.

(1.b) We list all elements in $\langle\langle Q\rangle\rangle$ and check whether some element is equal to $gvg^{-1}w^{-1}$ in $F(X)$.

(2) We list all homomorphisms $f : F(X)/\langle\langle Q\rangle\rangle \to F(S)/\langle\langle R\rangle\rangle$ and, given a homomorphism f, we check whether $f(w), f(v)$ are not conjugate in $\langle S|R\rangle$. Clearly if $f(w), f(v)$ are not conjugate in $\langle S|R\rangle$ then they are not conjugate in $\langle X|Q\rangle$. □

We remark that this proposition shows that the solvability of the word and the conjugacy problem is a property of the group and not of the presentation.

3.6 Residually finite groups, simple groups

Definition 3.5. A group G is called *residually finite* if for every $1 \neq g \in G$ there is a homomorphism φ from G to a finite group F such that $\varphi(g) \neq 1$.

Remark 3.6. A group G is residually finite if for every $g \in G, g \neq 1$ there is a finite index subgroup H of G such that $g \notin H$ (exercise).

If a group G is residually finite then clearly any subgroup of G is also residually finite.

Proposition 3.7. *Let G be a residually finite group and let $g_1, ..., g_n$ be distinct elements of G. Then there is a homomorphism $\varphi : G \to F$ where F is finite, such that $\varphi(g_i) \neq \varphi(g_j)$ for any $1 \leq i < j \leq n$.*

Proof. If $h_1, ..., h_k$ are non trivial elements of G there are homomorphisms $\varphi_i : G \to F_i$, where F_i are finite, such that $\varphi_i(h_i) \neq 1$ for every i. It follows that

$$\varphi = (\varphi_1, ..., \varphi_k) : G \to F_1 \times ... \times F_k$$

is a homomorphism to a finite group such that $\varphi(h_i) \neq 1$ for every i. Now we apply this observation to the set of non-trivial elements $g_i g_j^{-1}$ $(1 \leq i < j \leq n)$ and we obtain a homomorphism $\varphi : G \to F$ (F finite), such that $\varphi(g_i g_j^{-1}) \neq 1$, hence $\varphi(g_i) \neq \varphi(g_j)$ for any $1 \leq i < j \leq n$. □

Intuitively residually finite groups are groups that can be 'approximated' by finite groups.

Matrix groups furnish examples of residually finite groups. To whow this we will need two easy lemmas. We leave the proofs to the reader.

Lemma 3.1. *Let A, B be commutative rings with 1 and let $f : A \to B$ be a ring homomorphism. Then the map $\bar{f} : SL_n(A) \to SL_n(B)$ given by $\bar{f}((a_{ij})) = (f(a_{ij}))$ is a group homorphism.*

Lemma 3.2. *Let A be a subring of $\mathbb{Q}$. Assume that there is a prime p such that for any $a/b \in A$, p does not divide b. Then the map $\phi : A \to \mathbb{Z}_p$, $\phi(a/b) = ab^{-1}$ is a ring homomorphism.*

Proposition 3.8. *The group $\Gamma = GL_n(\mathbb{Z})$ is residually finite.*

Proof. Indeed, we take subgroups $\Gamma(p) \leqslant \Gamma$, $\Gamma(p) = \ker(\varphi_p)$, where $\varphi_p : \Gamma \to GL(n, \mathbb{Z}_p)$ is the reduction modulo p.

Assume $g \in \Gamma$ is a non-trivial element. If g has a non-zero off-diagonal entry $g_{ij} \neq 0$, then $g_{ij} \neq 0 \mod p$, whenever $p > |g_{ij}|$. Thus, $\varphi_p(g) \neq 1$.

If $g \in \Gamma$ has only zero entries off-diagonal then it is a diagonal matrix with only ± 1 on the diagonal, and at least one entry -1. Then $\varphi_3(g)$ has at least one 2 on the diagonal, hence $\varphi_3(g) \neq 1$.

Thus Γ is residually finite. □

Proposition 3.9. *Any finitely generated subgroup G of $SL_n(\mathbb{Q})$ (or $GL_n(\mathbb{Q})$) is a residually finite group.*

Proof. Let $G =< g_1, ..., g_n >$. Let $p_1, ..., p_k$ be the primes that appear in the denominators of the entries of the matrices $g_1^{\pm 1}, ..., g_n^{\pm 1}$. If p is any other prime then by lemmas 3.1, 3.2 we have a homomorphism:

$$\varphi_p : G \to SL_n(\mathbb{Z}_p)$$

Clearly for any $g \in G$, $g \neq 1$, $\varphi_p(g) \neq 1$ for some prime p.

Clearly the same holds for subroups of $GL_n(\mathbb{Q})$ as we may see $GL_n(\mathbb{Q})$ as a subgroup of $SL_{n+1}(\mathbb{Q})$.

□

By a similar argument one can show that the same proposition holds for any finitely generated subgroup of $GL_n(\mathbb{C})$. The same simple argument for $GL(n, \mathbb{Z})$ is a model for a proof of a harder theorem:

Theorem 3.2 (A. I. Mal'cev (1940)). *Let Γ be a finitely generated subgroup of $GL(n, R)$, where R is a commutative ring with unity. Then Γ is residually finite.*

Example 3.2. The group $(\mathbb{Q}, +)$ is not residually finite. Indeed if $f : \mathbb{Q} \to F$ is a homomorphism such that F is finite and $f(1) = g \neq 1$ then $g = f(1/n)^n$ for any n, which is clearly impossible in a finite group.

Exercise 3.6. Prove the following two statements:

1. Every subgroup of a residually finite group is residually finite.

2. Direct products of residually finite groups are again residually finite.

Lemma 3.3. *If a group G contains a residually finite subgroup of finite index, then G itself is residually finite.*

Proof. Let $H \leqslant G$ be a finite index residually finite subgroup. The intersection of all finite-index subgroups

$$\bigcap_{i \in I} H_i \tag{3.3}$$

of H is $\{1\}$. Since H has finite index in G and each $H_i \leqslant H$ as above has finite index in G, the intersection of all finite-index subgroups of G is contained in (3.3) and, hence, is trivial. □

Proposition 3.10. *A semidirect product of a finitely generated residually finite group with a (not necessarily finitely generated) residually finite group is also residually finite.*

Proof. Let G be a group that splits as a semidirect product $H \rtimes Q$, where H and Q are residually finite, and H is moreover finitely generated. Let p denote the projection homomorphism $G \to Q$.

Consider $g \in G \setminus \{1\}$. If g does not belong to H, then $p(g) \neq 1$ and the residual finiteness of Q implies that there exists a homomorphism of Q to a finite group which sends sends $p(g)$ to a non-trivial element. By composing the homomorphisms, we obtain a homomorphism of G to a finite group which sends g to a non-trivial element.

Suppose, therefore, that g is in H. Let $F < H$ be a finite-index subgroup which does not contain g. Since H is finitely generated, Proposition **??**, (2), implies that there exists a finite-index subgroup $A \leqslant F$ which is a characteristic subgroup of H. The subgroup $A \rtimes Q$ is a finite index subgroup in $G = H \rtimes Q$ that does not contain g. □

Theorem 3.3. *Let G be a residually finite group admitting a finite presentation $\langle S|R\rangle$. Then G has a solvable word problem.*

Proof. First, note that if Φ is a finite group, then it has solvable word problem: using the multiplication table in Φ we can "compute" every product of generators as an element of Φ and decide if this element is trivial or not.

Given a residually finite group G with finite presentation $\langle S|R\rangle$, and a word w in S, we run two procedures simultaneously:

(1) We list all the elements of $\langle\langle R\rangle\rangle$. This means: multiply conjugates of the relators $r_j \in R$ by products of $x_i \in X$ and their inverses, transform the product to a reduced word.

If we find that w equals one of the elements of $\langle\langle R\rangle\rangle$, then we stop and conclude that $g \equiv_G 1$.

(2) We list all homomorphism $\varphi : G \to S_n$, where S_n is the symmetric group on n letters ($n \in \mathbb{N}$): this procedure will try to send generators $x_1, \ldots, x_m$ of G to elements of S_m and then check if the images of the relators in G under this map are trivial or not.

For every such homomorphism, the procedure then checks if $\varphi(w) = 1$ or not. If it finds φ so that $\varphi(w) \neq 1$, then $w \in G$ is non-trivial and the process stops.

The point of residual finiteness is that, eventually, one of the procedures stops and we determine whether w is trivial or not. □

Theorem 3.4. *The free group F_n is residually finite.*

Proof. Since F_n is a subgroup of F_2 it is enough to show that F_2 is residually finite. One way to show this is to prove that F_2 is isomorphic to a subgroup of $GL_2(\mathbb{Z})$ (exercise). We give here a different proof. Let $w \in F_2 =< a, b >$ be a reduced word of length k. Let B be the set of reduced words of length less or equal to k. We consider the group of permutations of B, $Symm(B)$. We define now two permutations α, β of $Symm(B)$: If $|v| \leq k-1$ we define $\alpha(v) = av$ and we extend α to the words of length k in any way. Similarly if $|v| \leq k-1$ we define $\beta(v) = bv$ and we extend β in the words of length k in any way. We define now a homomorphism

$$\varphi : F_2 \to Symm(B),\ \varphi(a) = \alpha, \varphi(b) = \beta$$

Clearly $\varphi(w)(e) = w$ so $\varphi(w) \neq 1$. □

Definition 3.7. We say that a group G is *Hopf* if every epimorphism $\varphi : G \to G$ is 1-1.

Theorem 3.5. *If a finitely generated group G is residually finite then G is Hopf.*

Proof. Assume that G is residually finite but not Hopf. Let $f : G \to G$ be an onto homomorphism and let $1 \neq g \in ker\, f$. Let F be a finite group and let $\varphi : G \to F$ be a homomorphism such that $\varphi(g) \neq 1$.

Since f is onto there is a sequence $g_0 = g, g_1, ..., g_n, ...$ such that $f(g_n) = g_{n-1}$ for any $n \geq 1$. This implies that the homomorphisms

$$\varphi \circ f^{(n)} : G \to F$$

are all distinct since for any $n \geq 1$

$$\varphi \circ f^{(n)}(g_n) \neq 1, \ \varphi \circ f^{(n)}(g_k) = 1 \text{ for } k < n$$

This is a contradiction since G is finitely generated and so there are only finitely many homomorphisms from G to F. □

Corollary 3.1. *If A is a generating set of n elements of the free group of rank n, F_n, then A is a free basis of F_n.*

Proof. Let X be a free basis of F_n and let $\varphi : X \to A$ be a 1-1 function. Then φ extends to a homomorphism $\varphi : F_n \to F_n$. Since A is a generating set φ is onto. However F_n is residually finite and hence Hopf. It follows that φ is an isomorphism and A a free basis of F_n. □

Definition 3.8. A non-trivial group G is called *simple* if the only normal subgroups of G are $\{1\}$ and G.

Theorem 3.6. *Let $G = \langle S|R \rangle$ be a finitely presented simple group. Then G has a solvable word problem.*

Proof. Let w be a word in S. We remark that if $w \neq 1$ in G then $\langle\langle w \rangle\rangle = G$, so $\langle\langle w \cup R \rangle\rangle = F(S)$.

We enumerate in parallel the elements of $\langle\langle w \cup R \rangle\rangle$ and of $\langle\langle R \rangle\rangle$ in $F(S)$. If $w = 1$ then eventually w will appear in the list of $\langle\langle R \rangle\rangle$, while if $w \neq 1$ the set S will eventually appear in the list of $\langle\langle w \cup R \rangle\rangle$. □

Chapter 4

Group actions on Trees

4.1 Cayley graphs

Geometric methods were used in group theory since its inception, but it was M. Gromov in 1984 that set the foundations of modern group theory. His insight was that one can derive many algebraic properties of infinite groups from their 'geometry'. In fact looking at the geometry turned out to be very revealing of the group structure, more so than pure algebraic manipulations. This first section will explain what we mean by 'geometry' in this context. Riemannian geometry, even though it inspires many arguments that follow, is useless for studying finitely generated groups. Finitely generated groups are discrete objects with no interesting 'local' geometry. Their true geometry becomes apparent only 'at large scale'. Gromov's insight transformed the field, as by bringing geometry into play, other tools such as analysis, dynamics etc. became available for studying groups.

The oldest, and most common, way to 'geometrize' groups, is by their *Cayley graphs*. Indeed, every group may be turned into a geometric object (a graph) as follows. Given a group G and its generating set S, one defines the *Cayley graph* of G with respect to S. This is a directed graph $\Gamma = \Gamma(S, G)$ such that

- its set of vertices is G;
- its set of oriented edges is (g, gs), with $s \in S$.

We will use the notation $\overline{gh}$ and $[g, h]$ for the edge (g, h). In order to avoid confusion with the notation for the commutator of the elements g and h we will always add the word *edge* in this situation.

Usually, the underlying non-oriented graph $\widehat{\Gamma} = \widehat{\Gamma}(S, G)$ of Γ, i.e. the graph such that:

- its set of vertices is G;
- its set of edges consists of all pairs of elements in G, $\{g, h\}$, such that $h = gs$, with $s \in S$,

is also called the *Cayley graph of G with respect to S*.

More generally if $S \subset G$, where S is not necessarily a generating set we define the graph $\Gamma(S, G)$ as before to be the graph with vertices $\{g : g \in G\}$ and oriented edges $\{(g, gs) : g \in G,\ s \in S\}$.

Remark 4.1. The Cayley graph of G is a connected graph. Conversely, if $\widehat{\Gamma}(S, G)$ is connected for some $S \subset G$, then S is a generating set of G.

Definition 4.1. Recall that if v is a vertex of a non-oriented graph Γ we define the *valency* or the *degree* of v to be the number of edges incident with v. So $deg(v) = card\,\{e \in E(\Gamma) : o(e) = v\}$. We say that a graph Γ is *locally finite* if every vertex is incident to finitely many edges. A graph is called *regular* if all vertices have the same degree. A subgraph L of Γ is a *bi-infinite geodesic* if it is isometric to $\mathbb{R}$ (where we consider L to be equipped with the metric induced by Γ).

We remark that if $\widehat{\Gamma}$ is the Cayley graph of a finitely generated group with respect to a finite generating set then $\widehat{\Gamma}$ is a regular locally finite graph.

We endow the graph $\Gamma(S, G)$ with the standard length metric (where every edge has unit length). The restriction of this metric to G is called *the word metric associated to S* and it is denoted by dist_S or d_S.

Notation 4.2. For an element $g \in G$ and a generating set S we denote $\mathrm{dist}_S(1, g)$ by $|g|_S$, the *word norm* of g. With this notation, $\mathrm{dist}_S(g, h) = |g^{-1}h|_S = |h^{-1}g|_S$.

Convention 4.3. In this course, unless stated otherwise, all Cayley graphs are defined for finite generating sets S that do not contain 1 and are stable by inversion.

Much of the discussion in this section, though, remains valid for arbitrary generating sets, including infinite ones.

Remark 4.2. 1. Every group acts on itself, on the left, by left multiplication:

$$G \times G \to G\,,\ (g, h) \mapsto gh\,.$$

This action extends to any Cayley graph: if $[x, xs]$ is an edge of $\Gamma(S, G)$ with the vertices x, xs, we extend g to the isometry

$$g : [x, xs] \to [gx, gxs]$$

between the unit intervals. Both actions $G \curvearrowright G$ and $G \curvearrowright \Gamma(S, G)$ are by isometries.

Definition 4.4. Let G be a group acting on a space X. We say that G acts *freely* on X if for any $1 \neq g \in G$, $x \in X$, $gx \neq x$ (in other words all stabilizers are trivial).

Lemma 4.1. *A group* $G =< S >$ *acts freely on the set of vertices of* $\Gamma(S, G)$.

The group G *acts freely on* $\Gamma(S, G)$ *if and only if none of the generators is of order two.*

Proof. If $gx = x$ for some $x \in G$ then $g = 1$.

If there exists s of order 2 then s fixes the middle of the edge $[1, s]$.

Conversely, assume $g \in G \setminus \{1\}$ fixes a point x in $\Gamma(S, G)$. Then x cannot be a vertex and g fixes set-wise the unique edge containing x, and therefore it swaps its endpoints x, y. Thus $gx = y$ and $gy = x$, whence $g^2 x = x$ and $g^2 = 1$.

On the other hand $y = xs$, therefore $gx = xs$, and $s = x^{-1}gx$ is of order 2 as well. □

2. The action of the group on itself by right multiplication defines maps

$$R_g : G \to G\,,\; R_g(h) = hg$$

that are, in general, not isometries with respect to a word metric, but are at finite distance from the identity map:

$$\mathrm{dist}(\mathrm{id}(h), R_g(h)) = |g|_S\,.$$

Exercise 4.5. Prove that the word metric on a group G associated to a generating set S may also be defined

1. either as the unique maximal left-invariant metric on G such that

$$\mathrm{dist}(1, s) = \mathrm{dist}(1, s^{-1}) = 1\,,\; \forall s \in S\,;$$

2. or by the following formula: $\mathrm{dist}(g, h)$ is the length of the shortest word w in the alphabet $S \cup S^{-1}$ such that $w = g^{-1}h$ in G.

Below are two simple examples of Cayley graphs.

Exercise 4.1. Consider the group $\mathbb{Z}^2$ with the set of generators

$$S = \{a = (1, 0), b = (0, 1), a^{-1} = (-1, 0), b^{-1} = (0, -1)\}.$$

The Cayley graph $\Gamma(S, G)$ is the square grid in the Euclidean plane: the vertices are points with integer coordinates, two vertices are connected by an edge if and only if either their first or their second coordinates differ by ± 1. See Figure 4.1.

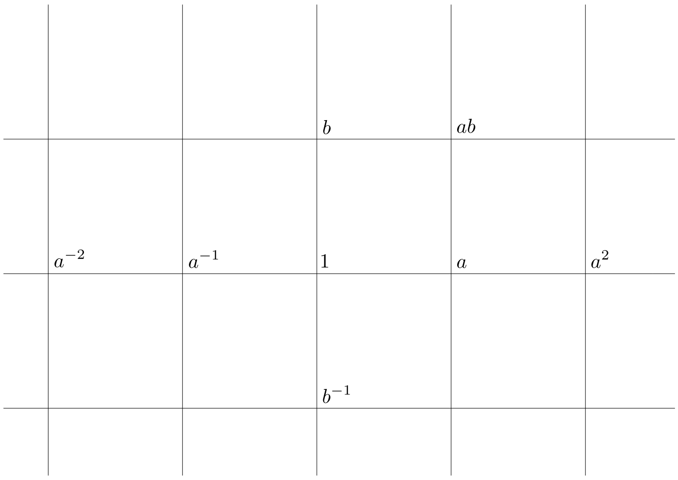

Figure 4.1: The Cayley graph of $\mathbb{Z}^2$.

The Cayley graph of $\mathbb{Z}^2$ with respect to the generating set$\{\pm(1,0), \pm(1,1)\}$ has the same set of vertices as the above, but the vertical lines are replaced by diagonal lines.

Exercise 4.2. Let G be the free group on two generators a, b. Take $S = \{a, b, a^{-1}, b^{-1}\}$. The Cayley graph $\Gamma(S, G)$ is the 4-valent tree (there are four edges incident to each vertex). See Figure 4.2.

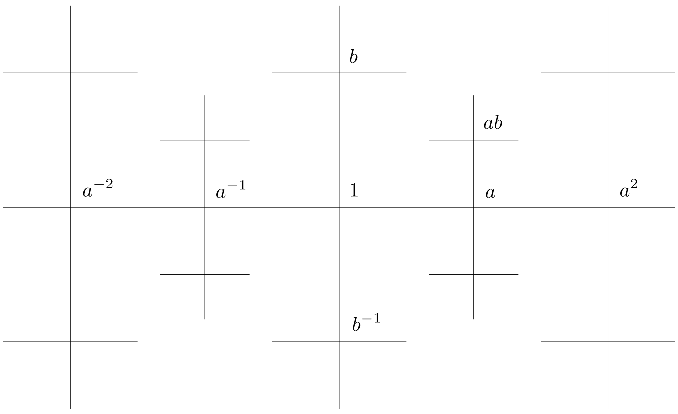

Figure 4.2: The Cayley graph of the free group F_2.

Thus, we succeeded in assigning to every finitely generated group G a metric space $\Gamma(S, G)$. The problem, however, is that this assignment

$$G \to \Gamma(S, G)$$

is far from canonical: different generating sets could yield completely different Cayley graphs.

Exercise 4.6. 1. Prove that if S and $\bar{S}$ are two finite generating sets of G, then the word metrics dist_S and $\mathrm{dist}_{\bar{S}}$ on G are bi-Lipschitz equivalent, i.e. there exists $L > 0$ such that

$$\frac{1}{L}\mathrm{dist}_S(g, g') \leqslant \mathrm{dist}_{\bar{S}}(g, g') \leqslant L\mathrm{dist}_S(g, g')\,, \forall g, g' \in G\,. \tag{4.1}$$

Hint: Verify the inequality (4.1) first for $g' = 1_G$ and $g \in S$; then verify the inequality for arbitrary $g \in G$ and $g' = 1_G$. Lastly, verify the inequality for all g, g' using left-invariance of word-metrics.

2. Prove that an isomorphism between two finitely generated groups is a bi-Lipschitz map when the two groups are endowed with word metrics.

Convention 4.7. From now on, unless otherwise stated, by a metric on a finitely generated group we mean a word metric coming from a finite generating set.

Exercise 4.8. Show that the Cayley graph of a finitely generated infinite group contains an isometric copy of $\mathbb{R}$, i.e. a bi-infinite geodesic. Hint: Apply Arzela-Ascoli theorem to a sequence of geodesic segments in the Cayley graph.

4.2 Actions of free groups on trees

Theorem 4.1. *Let S be a subset of a group G. The following are equivalent:*
i) $\widehat{\Gamma}(S, G)$ is a simplicial tree on which G acts freely.
ii) $S = X \sqcup X^{-1}$ and G is free with basis X.

Proof. ii) $\implies$ i).

Assume that G is free with basis X and $S = X \sqcup X^{-1}$. Every element of G can be represented by a reduced word on S, $s_1...s_n$. There is a path from 1 to $s_1...s_n$:

$$p = ((1, s_1), (s_1, s_1s_2), ..., (s_1s_2...s_{n-1}, s_1s_2...s_{n-1}s_n))$$

so X is connected. In general a reduced path starting at 1 corresponds to a reduced word on S, w. Since reduced words represent non trivial elements in G we have that $w \neq 1$ in G, so there are no circuits starting at 1. Since the action of G is transitive on vertices we deduce that X has no circuits, hence it is a tree.

i) $\implies$ ii) The group G acts freely on $\widehat{\Gamma}(S,G)$ therefore no element in S has order two. From every pair $\{s, s^{-1}\}$ in S we choose one element and we thus compose X such that $S = X \sqcup X^{-1}$.

Since $\widehat{\Gamma}(S,G)$ is connected there is a reduced path from 1 to any $g \in G$. Therefore any $g \in G$ can be written as a word on S. It follows that X generates G. Let $\varphi : F(X) \to G$ be the onto homomorphism defined by $\varphi(x) = x$ for all $x \in X$. Then if $s_1 ... s_n \in ker\, \varphi$ ($s_1 ... s_n$ reduced word) we have that the path

$$p = ((1, s_1), (s_1, s_1 s_2), ..., (s_1 s_2 ... s_{n-1}, s_1 s_2 ... s_{n-1} s_n))$$

is a reduced path in X from 1 to 1, which is impossible. We conclude that φ is 1-1, so $G \cong F(S)$.

□

Definition 4.9. Let G be a group acting on a graph X. We say that G acts on X *without inversions* if for every $g \in G$, $e \in E(X)$ we have that $ge \neq \bar{e}$.

Note that G acts *freely* on X if G acts on X without inversions and for any $1 \neq g \in G$, $v \in V(X)$, $gv \neq v$.

Theorem 4.2. *If a group G acts freely on a tree T then G is free.*

Proof.

Lemma 4.2. *There is a tree $X \subset T$ such that X contains exactly one vertex from each orbit of the action.*

Proof. Let X be a maximal subtree of T such that X contains at most one vertex from each orbit. Clearly such a tree exists by Zorn's lemma. Suppose that X does not intersect all orbits of vertices. Let v be a vertex of minimal distance from X such that X does not meet its orbit. If $d(v, X) = 1$ then we can add v to X contradicting its maximality. Otherwise if p is a reduced path from v to X and v' is the first vertex of p then $gv' \in X$ for some $g \in G$. But then $d(gv, X) = 1$ so we can add gv to X, a contradiction. We conclude that X contains exactly one vertex from each orbit. □

Let X be as in the lemma. We choose an orientation of the edges of T, $E^+ \subset E(T)$ such that E^+ is invariant under the action (that is $e \in E^+ \Rightarrow ge \in E^+$, for all $g \in G$). This is possible since the action is without inversions.

Consider the set

$$S = \{g \in G : \text{ there is an edge } e \in E^+ \text{ with } o(e) \in X,\ t(e) \in g(X)\}$$

We will show that G is a free group with basis S.

Clearly if $g_1 \neq g_2$ then $g_1X \cap g_2X = \emptyset$. Let T' be the tree that we obtain from T by contracting each translate gX to a point. Clearly G acts on T'. We will show that $T' \simeq \Gamma(S,G)$. We remark that $V(T') = \{gX : g \in G\}$, $E(T') = \{e \in T,\ e \notin GX\}$. The orientation of T induces an orientation of the edges of T' which we denote still by E^+. We define now $\varphi : T' \to \Gamma(S,G)$ as follows: $\varphi(gX) = g$. If $e \in E^+$ is an edge joining g_1X to g_2X then $s = g_1^{-1}g_2 \in S$ since $g_1^{-1}e$ joins X to $g_1^{-1}g_2X$. So we define $\varphi(e) = (g_1, g_1s) = (g_1, g_2)$. It is clear that φ is 1-1 and onto on the set of vertices $V(T')$. It is also onto on oriented edges: if (g, gs) is an oriented edge of $\Gamma(S,G)$ then there is an oriented edge $e \in T'$ joining X to sX and $\varphi(ge) = (g, gs)$. We note that if

$$\varphi(e_1) = \varphi(e_2) = (g, gs)$$

then e_1, e_2 are both oriented edges joining gX to gsX. But T' is a tree so $e_1 = e_2$ and φ is 1-1.

It follows that $\Gamma(S,G)$ is a tree, hence G is free (theorem 4.1).

□

Corollary 4.1. *Subgroups of free groups are free.*

Proof. Let $F(S)$ be a free group with basis S. Then $F(S)$ acts freely on its Cayley graph $\Gamma(S,G)$ which is a tree. So any subgroup H of $F(S)$ acts freely on $\Gamma(S,G)$ hence by the previous theorem H is free. □

4.3 Amalgams

The construction of amalgams allows us to 'combine' some given groups and construct new groups. Let A, B be two groups which have two isomorphic subgroups, that is there are embeddings $\alpha : H \to A$, $\beta : H \to B$. Intuitively the amalgam of A, B over H is a group that contains copies of A, B identified along H and with no other relation imposed. To simplify notation we pose $\alpha(h) = h$, $\beta(h) = \bar{h}$ for all $h \in H$.

One way to define amalgams is via their universal property:

Definition 4.10. We say that a group G is the *amalgamated product* of A, B over H and we write $G = A \underset{H}{*} B$ if there are homomorphisms $i_A : A \to G$, $i_B : B \to G$ which satisfy $i_A(h) = i_B(\bar{h})$, $\forall h \in H$, such that for every group L and homomorphisms $\alpha_1 : A \to L$, $\beta_1 : B \to L$ which satisfy $\alpha_1(h) = \beta_1(\bar{h})$, $\forall h \in H$, there is a unique homomorphism $\varphi : G \to L$ such that $\alpha_1 = \varphi \circ i_A$ and $\beta_1 = \varphi \circ i_B$.

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[dr, "\alpha_1"'] & G \arrow[d, dotted, "\varphi"'] & B \arrow[l, "i_B"'] \arrow[dl, "\beta_1"] \\
& L &
\end{tikzcd}$$

The amalgam of A, B over H depends of course on the maps α, β, it is however customary to suppress this on the notation. We note that it is not clear by the definition whether i_A, i_B are injective.

Remark 4.3. Assuming that an amalgam of A, B over H exists it is easy to see that this amalgam is unique using the universal property.

Indeed let G_1, G_2 be two such amalgams and let i_A, i_B, j_A, j_B be the inclusions of A, B in G_1, G_2 respectively. The homomorphisms j_A, j_B induce a homomorphism $j : G_1 \to G_2$ such that $j \circ i_A = j_A$, $j \circ i_B = j_B$. Similarly i_A, i_B induce a homomorphism $i : G_2 \to G_1$. The compositions of these maps induce homomorphisms $G_1 \to G_1$, $G_2 \to G_2$ which are both equal to the identity since they are induced by i_A, i_B and j_A, j_B respectively. So $G_1 \cong G_2$.

We show now that the amalgam of A, B over H exists:

Let $\langle S_1 | R_1 \rangle$, $\langle S_2 | R_2 \rangle$ be presentations of A, B respectively. Without loss of generality we assume that $S_1 \cap S_2 = \emptyset$. Then the amalgam of A, B over H is given by

$$A \underset{H}{*} B = \langle S_1 \cup S_2 | R_1 \cup R_2 \cup \{h = \bar{h} \, : \, h \in H\} \rangle$$

Indeed it is easy to see that this group satisfies the universal property of the definition.

When $H = \{1\}$ then the amalgam does not depend on the maps α, β and it is called *free product* of A, B; we denote this by $A * B$. We remark that $F_2 = \mathbb{Z} * \mathbb{Z}$. We would like to describe the elements of $A \underset{H}{*} B$ by 'words'. To simplify notation we identify H with its image in A, B. If $a \in A$ (or $b \in B$) we will denote the corresponding element of G by a (b) rather than $i_A(a)$ ($i_B(b)$). It is important to distinguish whether we see a as an element of A or of G since, a priori, it is possible that $a_1 = a_2$ in G while $a_1 \neq a_2$ in A (and similarly for B).

Let A_1, B_1 be sets of right coset representatives of H in A, B respectively, such that $1 \in A_1, 1 \in B_1$. So we have the 1-1 and onto maps:

$$H \times A_1 \to A, (h, a) \mapsto ha, \;\; H \times B_1 \to B, (h, b) \mapsto hb$$

A *reduced word* of the amalgam $A \underset{H}{*} B$ is a word of the form $(h, s_1, ..., s_n)$ where $h \in H$, $s_i \in A_1 \cup B_1$, $s_i \neq 1$ for every i and the s_i's alternate from A_1 to B_1. That is for all i, $s_i \in A_1 \implies s_{i+1} \in B_1$, $s_i \in B_1 \implies s_{i+1} \in A_1$. If $(h, s_1, ..., s_n)$ is a reduced word we associate to this the group element $hs_1...s_n \in A \underset{H}{*} B$. We say that the *length* of the reduced word $(h, s_1, ..., s_n)$ is n.

Theorem 4.3. *(Normal forms) Each $g \in G = A \underset{H}{*} B$ is represented by a unique reduced word.*

Proof. Any element $g \in G$ can be written as a product of the form

$$g = a_1 b_1 ... a_n b_n,\ a_i \in A, b_i \in B$$

By successive reductions we arrive at a reduced word, so we can represent g by a reduced word. We show now that this word is unique.

Let X be the set of all reduced words. We define an action of G on X. We recall that an action is a homomorphism $G \to Symm(X)$. By the universal property of the amalgam it is enough to define homomorphisms (actions) $A \to Symm(X)$, $B \to Symm(X)$ which agree on H. We define the action of A. If $a \in H$ and $(h, s_1, ..., s_n)$ is a reduced word we define

$$a \cdot (h, s_1, ..., s_n) = (ah, s_1, ..., s_n)$$

If $a \in A \setminus H$ and $(h, s_1, ..., s_n)$ a reduced word then there are two cases.
1st case: $s_1 \in B$. Then $ah = h_1 s$ for some $h_1 \in H$, $s \in A_1$ and we define

$$a \cdot (h, s_1, ..., s_n) = (h_1, s, s_1, ..., s_n)$$

2nd case: $s_1 \in A$. Then $ahs_1 = h_1 s$ for some $h_1 \in H$, $s \in A_1$. If $s \neq 1$ we define

$$a \cdot (h, s_1, ..., s_n) = (h_1, s, s_2, ..., s_n)$$

while if $s = 1$ we define

$$a \cdot (h, s_1, ..., s_n) = (h_1, s_2, ..., s_n)$$

One sees easily that if $a_1, a_2 \in A$ then

$$(a_1 a_2) \cdot (h, s_1, ..., s_n) = a_1 \cdot (a_2 \cdot (h, s_1, ..., s_n))$$

so we have indeed an action. We define the action of B similarly. So we have an action of G on X. Now if $g = hs_1...s_n$ where $(h, s_1, ..., s_n)$ is a reduced word then

$$g \cdot (1) = (h, s_1, ..., s_n)$$

It follows that the reduced word representing g is unique. $\square$

Corollary 4.2. *The homomorphisms* $i_A : A \to A \underset{H}{*} B$, $i_B : B \to A \underset{H}{*} B$ *are injective. So we can see* A, B *as subgroups of* $A \underset{H}{*} B$.

From now on we may identify elements of $A \underset{H}{*} B$ with reduced words.

Corollary 4.3. *Let* $A \underset{H}{*} B$ *be an amalgamated product. If* $(g_1, ..., g_n)$ *is such that* $g_i \in A \cup B$, $n > 1$, $g_i \notin H$ *for any* $i > 1$ *and the* g_i*'s alternate between* A *and* B *then* $g_1 g_2 ... g_n \neq 1$ *in* $A \underset{H}{*} B$.

Proof. Starting from g_n we replace successively the g_i's by elements of the form hs_i where s_i lies in $A_1 \cup B_1 \setminus 1$ (right coset representatives of H). Eventually we arrive at a reduced word representing $g_1 g_2 ... g_n$ which has length n if $g_1 \notin H$, and $n-1$ if $g_1 \in H$. It follows that $g_1 g_2 ... g_n \neq 1$. □

Exercise 4.3. Show that if $A \neq H \neq B$ then the center of $A \underset{H}{*} B$ is contained in H.

If $hs_1...s_n$ is a reduced word (element) in $A \underset{H}{*} B$ then we say that n is the length of this word. We say that a reduced element $hs_1...s_n$ $(n > 1)$ is *cyclically reduced* if $s_1 s_n$ is reduced.

Proposition 4.1. *1. Every element of* $A \underset{H}{*} B$ *is conjugate either to a cyclically reduced element or to an element of* A *or* B.

2. Every cyclically reduced element has infinite order.

Proof. 1. If $g = hs_1...s_n$ is not cyclically reduced then g is conjugate to an element of length $n-1$. We repeat till we arrive either at a reduced word or an element of A or B.

2. If g is cyclically reduced of length n then g^k has length kn so $g^k \neq 1$. □

Exercise 4.4. If K is a finite subgroup of $A \underset{H}{*} B$ then K is contained in a conjugate of either A or B.

Example 4.1. (Higman) Let

$$A = \langle a, s | sas^{-1} = a^2 \rangle$$

$$B = \langle b, t | tbt^{-1} = b^2 \rangle$$

Then $< a > \cong < b > \cong \mathbb{Z}$ so we may form the amalgam

$$G = A \underset{<a>=<b>}{*} B = \langle a, s, t | sas^{-1} = a^2,\ tat^{-1} = a^2 \rangle$$

The group G is not Hopf.

Proof. We define $\varphi : G \to G$ by

$$\varphi(a) = a^2,\ \varphi(s) = s,\ \varphi(t) = t$$

It is easy to see that the relations are satisfied so φ is a homomorphism. Moreover $\varphi(t^{-1}at) = t^{-1}a^2t = a$ so φ is onto. On the other hand $\varphi(s^{-1}ast^{-1}a^{-1}t) = s^{-1}a^2st^{-1}a^{-2}t = aa^{-1} = 1$. As $s^{-1}as \in A - <a>$, $t^{-1}a^{-1}t \in B - <b>$ (check this! see example 3.1) the element $(s^{-1}as)(t^{-1}a^{-1}t)$ has length 2 in the amalgam $A \underset{<a>=<b>}{*} B$ so $ker\,\varphi \neq 1$.

$\square$

4.4 Actions of amalgams on Trees

Definition 4.11. Let G be a group acting without inversions on a tree T. A subtree $S \subset T$ is called a *fundamental domain* of the action if the standard projection $p : S \to T/G$ is an isomorphism.

Theorem 4.4. *Let $G = A \underset{H}{*} B$ be an amalgamated product. Then G acts on a tree T with fundamental domain an edge $e = [P, Q]$ so that $stab(P) = A$, $stab(Q) = B$, $stab(e) = H$.*

Proof. We define the vertices of T to be

$$V(T) = G/A \sqcup G/B = \{gA : g \in G\} \sqcup \{gB : g \in G\}$$

and the edges

$$E(T) = G/H \sqcup \overline{G/H}$$

We define $o(gH) = gA$, $t(gH) = gB$. The action of G is the obvious one: If $g' \in G$ then

$$g' \cdot gA = (g'g)A, \quad g' \cdot gB = (g'g)B, \quad g' \cdot gH = (g'g)H$$

Clearly G acts transitively on the set of geometric edges of T and there are two orbits of vertices. T is connected since if $g = hs_1...s_n$, (reduced word of length n) then there is an edge joining gA to $hs_1...s_{n-1}B$ if $s_n \in A$. Otherwise there is an edge joining gB to $hs_1...s_{n-1}A$. Since gA, gB are joined by an edge we see by induction on the length of g that every vertex gA or gB can be joined by a path to $1 \cdot A$, so T is connected.

We note that if p a path starting and ending at $1 \cdot A$ then necessarily the length of p is even. Suppose now that p is a reduced path of length $2n$ starting at $1 \cdot A$. We claim that the vertices of p are of the form

$$1 \cdot A, a_1B, a_1b_1A, ..., a_1b_1...a_nb_nA$$

where $a_i \in A - H$ for $i > 1$ and $b_i \in B - H$ for all i. Indeed this is easily proven inductively as if e.g. $a_1b_1...a_kb_kA, gB$ are successive vertices then $gb = a_1b_1...a_kb_ka$ for some $a \in A, b \in B$. However $gbB = gB$ so we may denote the vertex gB by $a_1b_1...a_kb_kaB$ (in other words $a_{k+1} = a$). Note also that if $a \in H$ then $gB = a_1b_1...a_kB$ so the path is not reduced. It follows that the length of $a_1b_1...a_nb_n$ is at least $2n-1$ so $1A \neq a_1b_1...a_nb_nA$, ie there are no reduced paths starting and ending at A. Similarly there are no reduced paths starting and ending at B. As every vertex of T lies either in the orbit of A or of B we conclude that T has no circuits.

Therefore T is a tree. □

Corollary 4.4. *Let F be a subgroup of $A \underset{H}{*} B$ which intersects trivially any conjugate of A or B. Then F is free.*

Proof. Let T be the tree constructed in the theorem 4.4. The stabilizers of vertices of T are conjugates of A, B. Since F intersects trivially the conjugates of A, B, F acts freely on T. By theorem 4.2 F is a free group. □

Proposition 4.2. *Let $G = A*B$. Then the kernel of the natural map $\varphi : A*B \to A\times B$ is free.*

Proof. If $R = ker\,\varphi$ then R intersects trivially all conjugates of A, B since these map isomorphically to their image. By corollary 4.4 R is free. □

Corollary 4.5. *If A, B are finite groups then $A * B$ has a finite index subgroup which is free.*

Theorem 4.4 has a converse:

Theorem 4.5. *Assume that G acts on a tree T with fundamental domain an edge $e = [P, Q]$. If $stab(P) = A$, $stab(Q) = B$, $stab(e) = H$ then $G = A \underset{H}{*} B$.*

Proof. The inclusions $A \to G$, $B \to G$ induce a homomorphism

$$\varphi : A \underset{H}{*} B \to G$$

We consider the subgroup $G' =< A, B >$. Note that if for some $g_1 \in G$, $g_2 \in G'$ we have that $g_1P = g_2P$ then $g_2^{-1}g_1 \in A$ so $g_1 \in G'$. The same holds if $g_1Q = g_2Q$. So $(G - G')e \cap G'e = \emptyset$. On the other hand $T = Ge = (G - G')e \cup G'e$ and T is connected. Moreover the sets $(G - G')e, G'(e)$ are closed.

It follows that $G - G' = \emptyset$ and $G = G'$. Therefore φ is onto. We show now that φ is 1-1. Let $g = hs_1...s_n$ (reduced word in $A \underset{H}{*} B$) be an element of $ker\,\varphi$. Clearly $n > 1$.

We distinguish now two cases. If $s_n \in A$ then we see by induction on n that $d(gQ, Q) = n$ if n is even and $d(gQ, Q) = n + 1$ if n is odd. Similarly if $s_n \in B$ we see inductively that $d(gP, P) = n$ if n is even and $d(gP, P) = n + 1$ if n is odd. It follows that $g \neq 1$ in G so φ is 1-1.

$\square$

4.5 HNN extensions

Definition 4.12. Let G be a group, A a subgroup of G and $\theta : A \to G$ a monomorphism. The *HNN-extension* of G over A with respect to θ is the group

$$G \underset{A}{*} = \langle G, t | tat^{-1} = \theta(a), \, \forall a \in A \rangle = G* < t > / \langle\langle tat^{-1}\theta(a)^{-1}, a \in A \rangle\rangle$$

The letter t is called *stable letter* of the HNN-extension.

We remark that if $\langle S|R \rangle$ is a presentation of G then a presentation of $G \underset{A}{*}$ is given by

$$\langle S \cup \{t\} | R \cup \{tat^{-1} = \theta(a), \, \forall a \in A\} \rangle$$

Let A_1, A_2 be sets of right coset representatives of $A, \theta(A)$ in G so that $1 \in A_1, 1 \in A_2$. A *reduced word* of the HNN extension $G \underset{A}{*}$ is a word of the form

$$(g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, ..., t^{\epsilon_n}, g_n)$$

where $\epsilon_i = \pm 1$, $g_0 \in G$, $g_i \in A_1$ if $\epsilon_i = 1$, $g_i \in A_2$ if $\epsilon_i = -1$ and $g_i \neq 1$ if $\epsilon_{i+1} = -\epsilon_i$.

If $(g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n)$ is a reduced word we associate to this the group element $g_0 t^{\epsilon_1} ... t^{\epsilon_n} g_n \in G \underset{A}{*}$.

Theorem 4.6. *(Normal forms) Each $g \in G \underset{A}{*}$ is represented by a unique reduced word.*

Proof. It is easy to see by successive reductions that any $g \in G \underset{A}{*}$ can be represented by some reduced word. We show now that this representation is unique. We use a similar argument as for amalgamated products. Let X be the set of all reduced words. We define an action of $G \underset{A}{*}$ on X. To do this it is enough to define actions of G and $< t >$ and show that the relations are satisfied. Let $g \in G$ and $(g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n)$ a reduced word. We define

$$g \cdot (g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n) = (gg_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n)$$

Clearly this defines an action of G on X. We define now the action of t.

$$t\cdot(g_0,t^{\epsilon_1},...,t^{\epsilon_n},g_n)=\begin{cases}(\theta(a),t,g_0',t^{\epsilon_1},...,t^{\epsilon_n},g_n) & \text{if } g_0=ag_0',\ 1\neq g_0'\in A_1\\ (\theta(g_0),t,1,t^{\epsilon_1},...,t^{\epsilon_n},g_n) & \text{if } g_0\in A, \epsilon_1=1\\ (\theta(g_0)g_1,t^{\epsilon_2},...,t^{\epsilon_n},g_n) & \text{if } g_0\in A, \epsilon_1=-1\end{cases}$$

So t defines a 1-1 map $X\to X$. We show that this map is onto. If $(g_0,t^{\epsilon_1},...,t^{\epsilon_n},g_n)\in X$ then

$$(g_0,t^{\epsilon_1},...,t^{\epsilon_n},g_n)=\begin{cases}t\cdot(1,t^{-1},g_0,t^{\epsilon_1},...,t^{\epsilon_n},g_n) & \text{if } g_0\notin\theta(A)\\ t\cdot(ag_1,t^{\epsilon_2},...,t^{\epsilon_n},g_n) & \text{if } g_0=\theta(a),\ a\in A, \epsilon_1=1\\ t\cdot(a,t^{-1},1,t^{\epsilon_1}g_1,t^{\epsilon_2},...,t^{\epsilon_n},g_n) & \text{if } g_0=\theta(a),\ a\in A, \epsilon_1=-1\end{cases}$$

So t gives an element of $Symm(X)$. In other words we have defined homomorphisms $G\to Symm(X)$, $<t>\to Symm(X)$. It follows that there is an extension of these homomorphisms to $G*<t>\to Symm(X)$. We verify that tat^{-1} and $\theta(a)$ $(a\in A)$ act in the same way. So we have an action of $G\underset{A}{*}$ on X. If $g_0t^{\epsilon_1}...t^{\epsilon_n}g_n\in G\underset{A}{*}$ is an element corresponding to a reduced word then

$$g_0t^{\epsilon_1}...t^{\epsilon_n}g_n\cdot(1)=(g_0,t^{\epsilon_1},...,t^{\epsilon_n},g_n)$$

So each element is represented by a unique reduced word. □

Corollary 4.6. *The group G embeds in $G\underset{A}{*}$.*

Corollary 4.7. *Let $G\underset{A}{*}$ be an HNN extension. Let $(g_0,t^{\epsilon_1},g_1,t^{\epsilon_2},...,t^{\epsilon_n},g_n)$ be such that $g_i\in G$ for all i, $\epsilon_i=\pm1$, $g_i\notin A$ if $\epsilon_i=1$ and $\epsilon_{i+1}=-1$, $g_i\notin\theta(A)$ if $\epsilon_i=-1$ and $\epsilon_{i+1}=1$, then $g_0t^{\epsilon_1}g_1...t^{\epsilon_n}g_n\neq1$ in $G\underset{A}{*}$.*

Proof. Starting from g_n we replace successively the g_i's by elements of the form hs_i where s_i lies in $A_1\cup A_2$ (right coset representatives of $A,\theta(A)$). Specifically if after some successive reductions we need to rewrite $t^{\epsilon_i}g_i'$ we do the following: if $\epsilon_i=1$ we set $g_i'=hs_i$ with $h\in A, s_i\in A_1$. Then $t^{\epsilon_i}g_i'=\theta(h)t^{\epsilon_i}s_i$. If $\epsilon_i=-1$ we set $g_i'=\theta(h)hs_i$ with $h\in A, s_i\in A_2$. Then $t^{\epsilon_i}g_i'=ht^{\epsilon_i}s_i$. Note that g_{i-1} is replaced then by $g_{i-1}'=g_{i-1}\theta(h)$ in the first case and by $g_{i-1}'=g_{i-1}h$ in the second case. The hypothesis of the Corollary still holds for g_{i-1}' so we proceed to the next step. In particular the hypothesis of the corollary ensures that no successive letters t,t^{-1} or t^{-1},t appear as we apply this procedure, so we eventually arrive at a reduced word representing $g_0t^{\epsilon_1}g_1...t^{\epsilon_n}g_n$ which has length n. Hence $g_0t^{\epsilon_1}g_1...t^{\epsilon_n}g_n\neq1$. □

Definition 4.13. If a group G is an amalgam $G = A \underset{H}{*} B$ (with $A \neq H \neq B$) or an HNN-extension $G = A\underset{H}{*}$ then we say that G *splits* over H.

Example 4.2. (Higman, Neumann and Neumann) Any countable group embeds in a group with 2 generators.

Proof. Let $C = \{c_0 = e, c_1, c_2, ...\}$ be a countable group. We remark that the set of elements $S = \{a^n b a^{-n} : n \in \mathbb{N}\}$ forms a basis for free subgroup of the free group of rank 2, $F = F(a, b)$. Consider the group

$$H = F * C$$

The subgroups

$$A = \langle a^n b a^{-n} : n \in \mathbb{N} \rangle, \, B = \langle c_n b^n a b^{-n} : n \in \mathbb{N} \rangle$$

are both free of infinite rank by the normal form theorem for free products (theorem 4.3). Let $\phi : A \to B$ be the isomorphism given by $\phi(a^n b a^{-n}) = c_n b^n a b^{-n}$. Consider the HNN extension

$$G = H \underset{A}{*} = \langle H * < t > | t a^n b a^{-n} t^{-1} = c_n b^n a b^{-n}, \, \forall n \in \mathbb{N} \rangle$$

Clearly C embeds in G (normal form theorem for HNN extensions). Morover

$$t a^n b a^{-n} t^{-1} = c_n b^n a b^{-n} \implies c_n = t a^n b a^{-n} t^{-1} b^n a^{-1} b^{-n}$$

so G is generated by t, a, b, and in fact since $tbt^{-1} = a$, G is generated by a, t. $\square$

Chapter 5

Graphs of Groups

5.1 Fundamental groups of graphs of groups

Definition 5.1. Let G be a group acting without inversions on a graph X. We define the *quotient graph* of the action X/G as follows: If $v \in V(X)$, $e \in E(X)$ we set

$$[v] = \{gv : g \in G\}, \ \ [e] = \{ge : g \in G\}$$

The vertices and edges of the quotient graph are given by

$$V(X/G) = \{[v] : v \in V(X)\}, \ E(X/G) = \{[e] : e \in E(X)\}$$

and $o([e]) = [o(e)], \ t([e]) = [t(e)], \ \overline{[e]} = [\bar{e}]$.

We remark that since the action is without inversions $[\bar{e}] \neq [e]$. There is an obvious graph morphism

$$p : X \to X/G, \text{ given by } p(v) = [v], \ p(e) = [e], \ v \in V(X), e \in E(X)$$

Definition 5.2. A *graph of groups* (G, Y) consists of a connected graph Y and a map G such that

1. G assigns a group G_v to every vertex $v \in V(Y)$ and a group G_e to every edge $e \in E(Y)$, so that $G_e = G_{\bar{e}}$.
2. For each edge e there is a monomorphism $\alpha_e : G_e \to G_{t(e)}$.

Graphs of groups occur naturally in the context of group actions on trees. If a group G acts on a tree T without inversions then we can form the quotient graph $Y = T/G$ in the sense described above. We note that there is a projection $p : T \to T/G$.

To each vertex $v \in Y$ (or edge $e \in Y$) we associate a group G_v (G_e) where G_v is the stabilizer of a vertex in $p^{-1}(v)$ (edge in $p^{-1}(e)$). Note that all stabilizers of vertices in $p^{-1}(v)$ are isomorphic and the same holds for edges. If the vertex $v' \in p^{-1}(v)$ is an endpoint of the edge $e' \in p^{-1}(e)$ in T we have a monomorphism (inclusion) $stab(e') \to stab(v')$ and this is how we obtain the monomorphism $G_e \to G_v$. We will associate graphs of groups to actions more formally later, here we mention this as a source of examples and in order to put this definition in context.

Definition 5.3. The *path group* of the graph of groups (G, Y) is the group

$$F(G,Y) = \langle \mathop{*}_{v \in V(Y)} G_v \mathop{*}_{e \in E(Y)} \langle e \rangle | \bar{e} = e^{-1},\ e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g),\ \forall e \in E(Y),\ g \in G_e \rangle$$

If $G_v = \langle S_v | R_v \rangle$ then a presentation of $F(G, Y)$ is given by

$$\langle \bigcup_{v \in V(Y)} S_v \cup \{e \in E(Y)\} | \bigcup_{v \in V(Y)} R_v, \bar{e} = e^{-1},\ e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g),\ \forall e \in E(Y),\ g \in G_e \rangle$$

Remarks.

1. If $G_v = \{1\}$ for all $v \in V(Y)$ then $F(G, Y) = F(E^+(Y))$ (the free group with basis the geometric edges of Y).
2. If $G_e = \{1\}$ for all $e \in E(Y)$ then $F(G, Y) = \mathop{*}_{v \in V(Y)} G_v * F(E^+(Y))$.
3. There is an epimorphism $F(G, Y) \to F(E^+(Y))$ defined by sending all $g \in G_v$ (for all v) to 1.

Definition 5.4. A *path* c in the graph of groups (G, Y) is a sequence

$$c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$$

such that $t(e_i) = o(e_{i+1})$ and $g_i \in G_{o(e_{i+1})} = G_{t(e_i)}$ for all i. If

$$v_0 = o(e_1), v_1 = o(e_2) = t(e_1), ..., v_n = t(e_n)$$

we say that c is a path from v_0 to v_n and $(v_0, ..., v_n)$ is the sequence of vertices of the path c. We define $|c|$ to be the element of the path group: $|c| = g_0 e_1 g_1 ....e_n g_n$.

If $a_0, a_1 \in V(Y)$ we define

$$\pi[a_0, a_1] = \{|c| : c \text{ path from } a_0 \text{ to } a_1\}$$

If $a_0, a_1, a_2 \in V(Y)$ and $\gamma \in \pi[a_0, a_1]$, $\delta \in \pi[a_1, a_2]$ then $\gamma \cdot \delta \in \pi[a_0, a_2]$.

Proposition 5.1. *Let (G, Y) be a graph of groups. The set $\pi[a_0, a_0]$ ($a_0 \in V(Y)$) is a subgroup of $F(G, Y)$. We call this fundamental group of the graph of groups (G, Y) with base point a_0 and we denote it by $\pi_1(G, Y, a_0)$.*

Proof. It is enough to show that every element of $\pi[a_0, a_0]$ has an inverse in $\pi[a_0, a_0]$. If $c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$ is a path from a_0 to a_0 then

$$|c|^{-1} = g_n^{-1}\bar{e}_n....\bar{e}_1 g_0^{-1} \in \pi[a_0, a_0]$$

□

Definition 5.5. Let (G, Y) be a graph of groups and let T be a maximal tree of Y. We define the *fundamental group* of (G, Y) with respect to T, $\pi_1(G, Y, T)$ to be the quotient group

$$\pi_1(G, Y, T) = F(G, Y)/\langle\langle\{e, e \in T\}\rangle\rangle$$

We have the obvious quotient map $q : F(G, Y) \to \pi_1(G, Y, T)$.

Proposition 5.2. *The restriction of q to $\pi_1(G, Y, a_0)$ is an isomorphism, so*

$$\pi_1(G, Y, a_0) \cong \pi_1(G, Y, T)$$

Proof. We would like to define a homomorphism $f : \pi_1(G, Y, T) \to \pi_1(G, Y, a_0)$. Let $a \in V(Y)$ and $(e_1, ..., e_n)$ a geodesic path on T from a_0 to a. We set $g_a = e_1...e_n \in F(G, Y)$. If $a = a_0$ we set $g_a = 1$.

If e is an edge with $o(e) = a$, $t(e) = b$ we define

$$f(e) = g_a e g_b^{-1} \in \pi_1(G, Y, a_0)$$

Clearly if $e \in T$ then $f(e) = 1$ so this makes sense.

If $g \in G_a$ we define

$$f(g) = g_a g g_a^{-1} \in \pi_1(G, Y, a_0).$$

If e is an edge and $o(e) = P$, $t(e) = Q$ then

$$f(e\alpha_e(g)e^{-1}) = (g_P e g_Q^{-1})(g_Q \alpha_e(g) g_Q^{-1})(g_Q e g_P^{-1}) = g_P e \alpha_e(g) e^{-1} g_P^{-1} = g_P \alpha_{\bar{e}}(g) g_P^{-1}$$

and

$$f(\alpha_{\bar{e}}(g)) = g_P \alpha_{\bar{e}}(g) g_P^{-1}$$

so the relations are satisfied for all $e \in E(Y)$. It follows that f is a homomorphism.

Also $q \circ f(g) = g$ for all $g \in G_v$, $v \in V(T)$ and $q \circ f(e) = e$ for all $e \notin T$. So $q \circ f = id$.

We calculate now $f \circ q$. Let $(g_0, e_1, ..., e_n, g_n)$ be a path such that $g_0, g_n \in G_{a_0}$. If $e_i = [P_{i-1}, P_i]$ then $q(g_i) = g_i$ and $f(g_i) = g_{P_i} g_i g_{P_i}^{-1}$. Also $q(e_i) = e_i$ and $f(e_i) = g_{P_{i-1}} e_i g_{P_i}^{-1}$. We remark also that $g_{P_0} = g_{P_n} = g_{a_0} = 1$.

So

$$f \circ q(g_0 e_1 ... e_n g_n) = g_0 (e_1 g_{P_1}^{-1}) g_{P_1} ... g_{P_{n-1}}^{-1} (g_{P_{n-1}} e_n g_n) = g_0 e_1 ... e_n g_n$$

so $f \circ q = id$. □

Corollary 5.1. *The fundamental group of the graph of groups $\pi_1(G, Y, a_0)$ does not depend on the basepoint a_0.*

5.2 Reduced words

Definition 5.6. Let (G, Y) be a graph of groups and let $c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$ be a path. We say that c is *reduced* if:

1) $g_0 \neq 1$ if $n = 0$.

2) For every i if $e_{i+1} = \bar{e}_i$ then $g_i \notin \alpha_{e_i}(G_{e_i})$.

If c is a reduced path we say that $g_0 e_1 .... e_n g_n$ is a reduced word. We denote by $|c|$ the element of $F(G, Y)$ represented by the word $g_0 e_1 .... e_n g_n$.

Theorem 5.1. *If c is a reduced path then $|c| \neq 1$ in $F(G, Y)$. In particular for any vertex $v \in V(Y)$ the homomorphism $G_v \to F(G, Y)$ is injective.*

Proof. We prove first the theorem for finite graphs by induction on the number of edges. If Y is a single vertex there is nothing to prove. Otherwise we distinguish two cases:

Case 1: $Y = Y' \cup \{e\}$ where Y' is a connected graph and $v = t(e) \notin Y'$. In this case

$$F(G, Y) = (F(G, Y') * G_v) \underset{\alpha_e(G_e)}{*}$$

and a reduced word on $F(G, Y)$ corresponds to a reduced word in the HNN extension which is non trivial by corollary 4.7.

Case 2: $Y = Y' \cup \{e\}$ where Y' is a connected graph and $o(e), t(e) \in Y'$. In this case

$$F(G, Y) = F(G, Y') \underset{\alpha_e(G_e)}{*}$$

and a reduced word on $F(G, Y)$ corresponds to a word in the HNN extension which is non trivial by corollary 4.7.

This proves the theorem in case Y is finite. If Y is infinite and a reduced word w is equal to 1 in $F(G, Y)$ then it is equal to a product of finitely many conjugates of relators of $F(G,Y)$. However these relators involve only group elements and edge generators lying in a finite subgraph Y_1. By taking Y_1 big enough we may assume that the conjugating elements also lie in Y_1. It follows that $w = 1$ in $F(G, Y_1)$ which is a contradiction since w is a reduced word and Y_1 is finite.

□

Corollary 5.2. *For any vertex $v \in V(Y)$ the homomorphism $G_v \to \pi_1(G,Y,T)$ is injective.*

Proof. The homomorphism $G_v \to \pi_1(G,Y,v)$ is injective since $\pi_1(G,Y,v)$ is a subgroup of $F(G,Y)$ and if $1 \neq g \in G_v$ g is a reduced word in $F(G,Y)$ hence $g \neq 1$. However $\pi_1(G,Y,v) \cong \pi_1(G,Y,T)$ and $g \in G_v$ maps to itself in $\pi_1(G,Y,T)$ so $g \neq 1$ in $\pi_1(G,Y,T)$.

□

Remark 5.1. If Y consists of a single edge $e = [u,v]$ with $u \neq v$ then one sees from the presentation that $\pi_1(G,Y,T) = G_u \underset{G_e}{*} G_v$. If the endpoints of e are equal ($u = v$) then $\pi_1(G,Y,T) = G_v \underset{\alpha_e(G_e)}{*}$ where the homomorphism of the HNN extension $\theta : \alpha_e(G_e) \to G_v$ is given by $\theta(g) = \alpha_{\bar{e}} \circ \alpha_e^{-1}$ and the stable letter is e.

In general if $Y = Y' \cup e$ and $v = t(e) \notin Y'$ then

$$\pi_1(G,Y,T) = \pi_1(G,Y',T') \underset{G_e}{*} G_v$$

while if $v = t(e) \in Y'$ then

$$\pi_1(G,Y,T) = \pi_1(G,Y',T) \underset{\alpha_e(G_e)}{*}$$

As we did for amalgams and HNN-extensions we can find a set of words that is in one to one correspondence with the elements of the fundamental group of the graph of groups.

Let (G,Y) be a graph of groups. For each edge $e \in E(Y)$ we pick a set S_e of left coset representatives of $\alpha_{\bar{e}}(G_e)$ in $G_{o(e)}$. We require that $1 \in S_e$.

Definition 5.7. We say that the path $(s_1, e_1, ..., s_n, e_n, g)$ is *S-reduced* if $s_i \in S_{e_i}$ for all i, $s_i \neq 1$ if $e_{i-1} = \bar{e}_i$ and $g \in G_{t(e_n)}$.

Lemma 5.1. *Let $a, b \in V(Y)$. Then every element of $\pi[a,b]$ is represented by a unique S-reduced path.*

Proof. Existence. For every element $\gamma \in \pi[a,b]$ there is a reduced path $c = (g_1, e_1, g_2, e_2, ..., g_n, e_n, g)$ such that $\gamma = |c|$. We can write $g_1 = s_1 \alpha_{\bar{e}_1}(h_1),\ s_1 \in S_{e_1},\ h_1 \in G_{e_1}$. So

$$g_1 e_1 = s_1 \alpha_{\bar{e}_1}(h_1) e_1 = s_1 e_1 \bar{e}_1 \alpha_{\bar{e}_1}(h_1) e_1 = s_1 e_1 \alpha_{e_1}(h_1)$$

So we replace c by $(s_1, e_1, \alpha_{e_1}(h_1) g_2, e_2, ..., e_n, g_n)$ and we continue similarly replacing $\alpha_{e_1}(h_1) g_2$ and so on till we arrive at an S-reduced path c' such that $|c'| = \gamma$.

Uniqueness. Let

$$c = (s_1, e_1, ..., s_n, e_n, g),\ c' = (t_1, y_1, ..., t_k, y_k, h)$$

be S-reduced paths such that $|c| = |c'|$. Then

$$s_1 e_1 .... s_n e_n g = t_1 y_1 .... t_k y_k h \Rightarrow h^{-1} y_k^{-1} ... y_1^{-1} t_1^{-1} s_1 e_1 .... s_n e_n g = 1$$

Obviously this word is not reduced so $y_1 = e_1$ and $t_1^{-1} s_1 \in \alpha_{\bar{e}_1}(G_{e_1})$. Since t_1, s_1 are left coset representatives of $\alpha_{\bar{e}_1}(G_{e_1})$ we have $t_1 = s_1$. So $y_1^{-1} t_1^{-1} s_1 e_1 = 1$. Continuing in the same way we see that all corresponding elements are equal so $c = c'$.

□

5.3 Graphs of groups and actions on Trees

Let (G, Y) be a graph of groups. We will show in this section that the fundamental group of this graph of groups acts on a tree T so that the quotient graph of this action is isomorphic to Y.

The construction of T resembles the construction of the universal cover in topology. The universal cover $\tilde{X}$ of a space X is defined using the paths of X modulo an equivalence relation (homotopy). Here we do something similar: we consider paths in the graph of groups. The group elements on the paths account for the branching of the tree. A trivial case which illustrates this point is the case of a $\mathbb{Z}_2$ action on a tree with 2 edges fixing the vertex in the middle and permuting the 2 edges. The quotient space is just a single edge, so topologically it is the universal cover of itself. However we can recover the original 2-edge tree using the $\mathbb{Z}_2$ stabilizer of the middle vertex.

Let $a_0 \in V(Y)$. We consider the set of paths in (G, Y):

$$\pi[a_0, a] = \{|c| : c \text{ path from } a_0 \text{ to } a\}$$

We define an equivalence relation in $\pi[a_0, a]$: $|c_1| \sim |c_2|$ if $|c_1| = |c_2| g$ for some $g \in G_a$. We define then

$$V(T) = \bigcup_{a \in V(Y)} \pi[a_0, a]/\sim$$

We remark that an element of $\pi[a_0, a]/\sim$ corresponds to a unique S-reduced path of the form: $(s_1, e_1, ..., s_n, e_n)$ where $t(e_n) = a$ and $o(e_1) = a_0$. Indeed note that

$$|(s_1, e_1, ..., s_n, e_n)| \sim |(s_1, e_1, ..., s_n, e_n, g)| \ \ (g \in G_a)$$

So we may identify the vertices of T with S-reduced paths of the form $(s_1, e_1, ..., s_n, e_n)$. A geometric edge of T now is given by a pair of S-reduced paths that differ by an edge of Y:

$$\{(s_1, e_1, ..., s_n, e_n),\ (s_1, e_1, ..., s_n, e_n, s_{n+1}, e_{n+1})\}$$

Clearly T is connected since (1) can be joined to any other vertex by a path. By our definition of edges a reduced path in T from (1) to a vertex $v \in V(T)$ corresponds to an S-reduced path. However, it follows from lemma 5.1 that if $v \in V(T)$ there is a unique S-reduced path joining $(1), v$. Therefore T is a tree.

We define now the action of $H = \pi_1(G, Y, a_0) = \pi[a_0, a_0]$ on T. If $g \in \pi[a_0, a_0]$ and $v \in \pi[a_0, a]$ then $gv \in \pi[a_0, a]$. So we define $g \cdot [v] = [gv]$ (where we denote by $[v]$ the equivalence class of v in $\pi[a_0, a]/\sim$). This defines an action of H on $V(T)$ since $(g_1g_2) \cdot [v] = g_1 \cdot (g_2 \cdot [v])$. We note that adjacent vertices go to adjacent vertices under this action so we have an action on T. We remark that if $v_1, v_2 \in \pi[a_0, a]$ then $v_2v_1^{-1} \in \pi[a_0, a_0]$ and $(v_2v_1^{-1})\cdot[v_1] = [v_2]$. It follows that we can identify the vertices of the quotient graph T/H with the vertices of Y. We show now that the edges of the quotient graph T/H correspond to the edges of Y too. Let $e_1 = ([v], [vs_1e]), e_2 = ([v], [vs_2e])$ be two edges of T with $o(e_1) = o(e_2) = [v]$, $s_1, s_2 \in G_{o(e)}$. If $g = v(s_2s_1^{-1})v^{-1}$ we have that $g \in \pi[a_0, a_0]$ and $g \cdot e_1 = e_2$. So both edges lie in the same orbit and this orbit corresponds to the edge $e \in E(Y)$.

We can see further that stabilizers of vertices and edges of T are conjugates of vertex and edge groups of (G, Y).

First we prove a conjugation relation in the path group:

Proposition 5.3. *1. If $[v] \in V(T)$ and $v \in \pi[a_0, b]$ then $stab([v]) = vG_bv^{-1}$.*

2. If $\delta \in E(T)$, $\delta = [[v], [vge]]$ where $e = [a, b]$, $g \in G_a$ then $stab(\delta) = (vg)(\alpha_{\bar{e}}(G_e))(vg)^{-1}$.

Proof. 1. Clearly $vG_bv^{-1} \subset stab([v])$. Assume now that $g \in stab([v])$. Then by the definition of $V(T)$ $gv = vg_b$, $g_b \in G_b$. So $g \in vG_bv^{-1}$. We conclude that $stab([v]) = vG_bv^{-1}$.

2. $stab(\delta) = stab([v]) \cap stab([vge])$. So

$$stab(\delta) = vG_av^{-1} \cap (vge)G_b(vge)^{-1} = v(G_a \cap geG_be^{-1}g^{-1})v^{-1} =$$

$$= (vg)(G_a \cap eG_be^{-1})(vg)^{-1}$$

since $g \in G_a$. We remark that $eG_be^{-1} \cap G_a = \alpha_{\bar{e}}(G_e)$. This is because if $g_b \in G_b$, either eg_be^{-1} is a reduced word and so does not lie in G_a or $g_b \in \alpha_e(G_e)$ and then $eg_be^{-1} \in G_a$. We conclude that

$$stab(\delta) = (vg)(\alpha_{\bar{e}}(G_e)(vg)^{-1}$$

$\square$

Then we prove that the conjugation relation can be changed into a conjugation in $H = \pi_1(G, Y, a_0)$. In order to do this, we must identify all the vertex groups with subgroups in H. As explained in Section 5.1, it suffices to

- choose a maximal subtree $T_Y \subset Y$, and set $g_a = e_1 \dots e_n$ the unique geodesic path in T_Y from a_0 to a vertex a.
- $\forall g \in G_a$, identify it with $\hat{g} = g_a g g_a^{-1}$. Let $\hat{G}_a$ be the image of G_a, for every vertex a in Y.
- for every edge e with $o(e) = a$, the edge subgroup $\alpha_{\bar{e}}(G_e)$ of G_a appears as a subgroup $\hat{G}_e$ of H via the map $g \mapsto \hat{g} = g_a g g_a^{-1}$.

We can then reformulate the previous proposition as follows.

Proposition 5.4. *1. If $[v] \in V(T)$ and $v \in \pi[a_0, b]$ then*

$$\mathrm{Stab}([v]) = h\hat{G}_b h^{-1},$$

where $h = vg_b^{-1} \in H = \pi_1(G, Y, a_0)$.

2. If $\delta \in E(T)$, $\delta = [[v], [vge]]$ where $e = [a, b]$, $g \in G_a$ then

$$\mathrm{Stab}(\delta) = vgg_a^{-1}\hat{G}_e g_a g^{-1} v^{-1} = h\hat{G}_e h^{-1},$$

with $h = vgg_a^{-1} \in H$.

We denote the tree T by $(\widetilde{G, Y, a_0})$ and we say that it is the *universal covering tree* or *the Bass-Serre tree* of the graph of groups (G, Y).

5.4 Quotient graphs of groups

We showed in the previous section that if $\pi_1(G, Y, a_0)$ is the fundamental group of a graph of groups then $\pi_1(G, Y, a_0)$ acts on a tree T ,without inversions, with quotient graph Y. The converse is also true: If a group Γ acts on a tree T with quotient Y, then there is a graph of groups (G, Y) so that $\pi_1(G, Y, a_0) = \Gamma$.

We explain now how to associate a graph of groups (G, Y) to an action $\Gamma \curvearrowright T$ (where T is a tree). We define $Y = T/\Gamma$. We have the projection map $p : T \to Y$. Let $X \subset S \subset T$ be subtrees of T such that $p(X)$ is a maximal tree of Y, $p(S) = Y$ and the map p restricted to S is 1-1 on the set of edges. We introduce some convenient notation: if v, e are respectively a vertex and an edge of Y we write v^X for the vertex of X for which $p(v^X) = v$ and e^S for the edge of S for which $p(e^S) = e$. We define now a graph of groups with Y as underlying graph. If $v \in V(Y)$ we set $G_v = stab(v^X)$. If $e \in E(Y)$ we set $G_e = stab(e^S)$. It remains to define monomorphisms $\alpha_e : G_e \to G_{t(e)}$. For every $x \in V(S)$ we pick $g_x \in \Gamma$ such that $g_x x \in X$. If $x \in X$ we take $g_x = 1$. If $x = t(e^S)$ we define:

$$\alpha_e : G_e \to G_{t(e)}, \text{ by } \alpha_e(g) = g_x g g_x^{-1}$$

In this way we define a graph of groups (G, Y). We define a homomorphism $\varphi : F(G, Y) \to \Gamma$ as follows: $\varphi|_{G_a} = id$ for all $a \in V(Y)$. If $e \in E(Y)$ and $y = o(e^S)$, $x = t(e^S)$ then we define $\varphi(e) = g_y g_x^{-1}$. We verify that the relations are satisfied:

$$\varphi(e\alpha_e(g)e^{-1}) = (g_y g_x^{-1})(g_x g g_x^{-1})(g_y g_x^{-1})^{-1} = g_y g g_y^{-1}$$

and

$$\varphi(\alpha_{\bar{e}}(g)) = g_y g g_y^{-1}$$

So φ is indeed a homomorphism. We note that if $e \in p(X)$ then $\varphi(e) = 1$ so we have in fact a homomorphism

$$\varphi : \pi_1(G, Y, p(X)) = \pi_1(G, Y, a_0) \to \Gamma$$

We have the following:

Theorem 5.2. *The map $\varphi := \pi_1(G, Y, a_0) \to \Gamma$ is an isomorphism. If $\tilde{T}$ is the universal covering tree of (G, Y) then there is a graph morphism $\psi : \tilde{T} \to T$ such that ψ is 1-1 and onto and $\psi(gv) = \varphi(g)\psi(v)$ for all $v \in V(\tilde{T})$, $g \in \pi_1(G, Y, a_0)$.*

We omit the proof of this theorem. What this theorem essentially says is that we can recover the group and the action on the tree by the quotient graph of groups.

We can now understand subgroups of fundamental groups of graphs of groups.

Theorem 5.3. *Let $\Gamma = \pi_1(G, Y, a_0)$ where (G, Y) is a graph of groups. If B is a subgroup of Γ then there is a graph of groups (H, Z) such that $B = \pi_1(H, Z, b_0)$ and for every $v \in V(Z)$, $e \in E(Z)$, $H_v \leq gG_a g^{-1}$, $H_e \leq \gamma G_y \gamma^{-1}$ for some $a \in V(Y)$, $y \in E(Y)$ and $g, \gamma \in \Gamma$.*

Proof. Γ acts on a tree T with quotient graph of groups (G, Y). Since $B \leq \Gamma$, B acts also on T and the vertex and edge stabilizers of B are contained in the vertex and edge stabilizers of Γ. If $Z = T/B$ it is clear that the quotient graph of groups (H, Z) that we obtain from the action of B satisfy the assertions of the theorem. $\square$

Corollary 5.3. *(Kurosh's theorem) Let $G = G_1 * ... * G_n$. If $H \leq G$ then $H = (\underset{i \in I}{*} H_i) * F$ where F is a free group and the H_i's are subgroups of conjugates of the G_j's.*

Proof. G is the fundamental group of a graph of groups with underlying graph a tree with n vertices labeled by $G_1, ..., G_n$ and trivial edge groups. We apply now the previous theorem. $\square$

We mention two important theorems on the structure of finitely presented groups.

We say that a group G is *indecomposable* if it can not be written as a non-trivial free product $G = A * B$.

Theorem 5.4. *(Grushko) Let G be a finitely generated group. There are finitely many indecomposable groups $G_1, ..., G_k$ and $n \geq 0$ such that*

$$G = G_1 * ... * G_k * \mathbb{F}_n$$

Moreover if we have another decomposition of G as

$$G = H_1 * ... * H_m * \mathbb{F}_r$$

where H_i are indecomposable then $m = k$, $r = n$, and after reordering H_i is conjugate to G_i for all i.

Theorem 5.5. *(Dunwoody) Let Γ be a finitely presented group. Then Γ can be written as $\Gamma = \pi_1(G, Y, a_0)$ where (G, Y) is a finite graph of groups such that all edge groups are finite and all vertex groups do not split over finite groups.*

Dunwoody has shown that this last theorem does not generalize to all finitely generated groups.

Chapter 6

Groups as geometric objects

One of the most convincing demonstrations of the geometric point of view is the theory of hyperbolic groups. This is a class of groups which is generic (in a precise statistical sense 'most' groups are hyperbolic) and which can be studied by geometric methods. The theory of hyperbolic groups unifies the small cancellation theory which has algebraic origin and the deep theory of negatively curved manifolds. We will show in the following sections that the word and conjugacy problem are solvable for hyperbolic groups and we will give an introduction to the geometric tools used to study them.

6.1 Quasi-isometries

Definition 6.1. 1. A (usually non-continuous) map between metric spaces $f : X \to Y$ is called

- a (K, A)-*quasi-isometric embedding*, for some constants $K \geq 1, A > 0$, if for all $x_1, x_2 \in X$
$$\frac{1}{K}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Kd(x_1, x_2) + A;$$
- a (K, A)-*quasi-isometry* if moreover for all $y \in Y$ there is some $x \in X$ such that $d(y, f(x)) \leq A$.

2. A (K, A)-quasi-isometric embedding $q : I \to X$, where $I = [a, b]$ is an interval in $\mathbb{R}$ and X is a metric space is called a (K, A)-*quasi-geodesic*. The points $x = q(a)$ and $y = q(b)$ are called *its endpoints*. We also say that q *joins* x and y.

3. When there exists a quasi-isometry $f : X \to Y$ we say that X, Y are *quasi-isometric* and we write $X \sim Y$.

Example 6.1. 1. $\mathbb{R}^n$ and $\mathbb{Z}^n$ are quasi-isometric.

2. Any metric space of finite diameter is quasi-isometric to a point.

Exercises 6.1. 1. Show that $\sim$ is an equivalence relation.

2. Let S_1, S_2 be finite generating sets of a group G. Show that $\Gamma(S_1, G) \sim \Gamma(S_2, G)$.

3. Let T_3, T_4 be the regular trees of degrees, respectively, 3,4. Show that T_3, T_4 are quasi-isometric.

Given $\epsilon, \delta > 0$ a subset N of a metric space X is called an (ϵ, δ)-*net* (or simply a net) if for every $x \in X$ there is some $n \in N$ such that $d(x, n) \leq \epsilon$ and for every $n_1, n_2 \in N$, $d(n_1, n_2) \geq \delta$.

A set N that satisfies only the second condition (i.e. for every $n_1, n_2 \in N$, $d(n_1, n_2) \geq \delta$) is called δ-separated.

Exercises 6.2. 1. Show that any metric space X has a $(1, 1)$-net.

2. Show that if $N \subset X$ is a net then $X \sim N$.

3. Show that $X \sim Y$ if and only if there are nets $N_1 \subset X, N_2 \subset Y$ and a bilipschitz map $f : N_1 \to N_2$ (i.e. f is a bijection and f, f^{-1} are both Lipschitz).

4. Give an example of a metric space which is not quasi-isometric to any graph.

5. Let G be a finitely generated group. Show that $H < G$ is a net in G if and only if H is a finite index subgroup of G.

It turns out that if a finitely generated group acts 'nicely' on a 'nice' metric space then the space is quasi-isometric to the group.

We make this precise below.

Definition 6.2. Let $p : [0, 1] \to X$ be a path in a metric space (X, d). We define the *length* of p to be the supremum of

$$\sum_{i=0}^{n} d(p(t_i), p(t_{i+1}))$$

over all partitions $0 = t_0 < t_1 < ... < t_n = 1$ $(n \in \mathbb{N})$ of $[0, 1]$.

Definition 6.3. We say that X is a *geodesic metric space* if for any $a, b \in X$ there is a path p joining a, b such that $length(p) = d(a, b)$. Such a path p is called *geodesic*.

It will be convenient to parametrize paths with respect to arc-length. We recall that a path $p : [0, l] \to X$ is said to be parametrized by arc-length if

$$|t - s| = length(p([t, s])),\ \forall t, s \in [a, b]$$

If X is a geodesic metric space and $a, b \in X$ we denote by $[a, b]$ a geodesic path joining them.

Examples. 1. Connected graphs with the metric defined earlier are geodesic metric spaces.

2. $\mathbb{R}^n$ with the Euclidean distance and, more generally, complete Riemannian manifolds are geodesic metric spaces (Hopf-Rinow).

3. $\mathbb{R}^2 - \{(0,0)\}$ is not a geodesic metric space.

Exercise 6.1. Prove that for every $K \geq 1$ and $A \geq 0$ there exists $\lambda \geq 1$, $\mu \geq 0$ and $D \geq 0$ such that the following is true. Given an (K, A)-quasi-geodesic $q : I \to X$ of endpoints x, y in a geodesic metric space X there exists a (continuous) path $\alpha : I' \to X$ of endpoints x, y such that:

- for all $t, s \in I$,
$$\mathit{length}\,(\alpha([t,s])) \leq \lambda d(\alpha(t), \alpha(s)) + \mu;$$
- for every $x \in I$, $d(q(x), \alpha(I')) \leq D$;
- for every $t \in I'$, $d(\alpha(t), q(I)) \leq D$.

Definition 6.4. We say that a metric space X is *proper* if every closed ball in X is compact.

Example 6.2. A graph with a vertex of infinite degree is not a proper metric space.

Definition 6.5. Assume that a group G acts on a metric space X by isometries. We say that the action is *co-compact* if there is a compact $K \subset X$ such that

$$\bigcup_{g \in G} \{gK\} = X$$

We say that G acts *properly discontinuously* on X if for every compact $K \subset X$ the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite.

Theorem 6.1. *(Milnor-Svarč lemma) Let X be a proper geodesic metric space. If G acts by isometries, properly discontinuously and co-compactly on X then:*

1) G is finitely generated.

2) If S is a finite generating set of G the map

$$f : \Gamma(S, G) \to X, \;\; g \mapsto gx_0$$

is a quasi-isometry (for any fixed $x_0 \in X$).

Proof. Let $R > 0$ be such that the G-translates of $B = B(x_0, R)$ cover X, i.e.

$$\bigcup_{g \in G} \{gB\} = X$$

The set

$$S = \{s \in G : d(sx_0, x_0) \leq 2R + 1\}$$

is finite since the action of G is properly discontinuous. We claim that S is a generating set of G. Indeed let $g \in G$. Consider a geodesic path $[x_0, gx_0]$. If

$$k - 1 < d(x_0, gx_0) \leq k, \ (k \in \mathbb{N})$$

consider $x_1, ..., x_k = gx_0$ such that $d(x_i, x_{i+1}) \leq 1$ for all $i = 0, ..., k-1$. Pick $g_i \in G$, $i = 1, ..., k-1$ such that $d(g_i x_0, x_i) \leq R$. Then $d(g_i x_0, g_{i+1} x_0) \leq 2R + 1$ so $g_i^{-1} g_{i+1} \in S$. We pick $g_0 = e, g_k = g$. We have then

$$g = g_k = (eg_1)(g_1^{-1} g_2)...(g_{k-2}^{-1} g_{k-1})(g_{k-1}^{-1} g_k)$$

So g can be written as a product of elements in S.

Let's denote now by d_S the distance in $\Gamma(S, G)$. The previous calculation shows that

$$d(gx_0, x_0) \geq d_S(g, e) - 1 \qquad (*)$$

Assume that $d_S(g, e) = n$, so $g = s_1 ... s_n$ where $s_i \in S \cup S^{-1}$ for all i. Then

$$d(gx_0, x_0) = d(s_1 ... s_n x_0, x_0) \leq d(s_1 ... s_n x_0, s_1 ... s_{n-1} x_0) + ... + d(s_1 x_0, x_0) \leq (2R + 1)n$$

So

$$d(gx_0, x_0) \leq (2R + 1) d_S(g, e) \qquad (**)$$

As $d_S(g, h) = d_S(h^{-1} g, e)$ and $d(gx_0, hx_0) = d(h^{-1} g x_0, x_0)$, it follows by by $(*), (**)$ that the map $g \to gx_0$ is a quasi-isometry between $\Gamma(S, G)$ and Gx_0. Since for any $x \in X$ there is some gx_0 with $d(x, gx_0) \leq R$, f is a quasi-isometry from G to X.

□

Corollary 6.1. *1. Let $G = < S >$ be a finitely generated group and let H be a finite index subgroup of G. Then H is quasi-isometric to G.*

2. Let G be a finitely generated group and let N be a finite normal subgroup of G. Then G/N is quasi-isometric to G.

Proof. 1. H acts freely and co-compactly on $\Gamma(S, G)$.

2. G acts properly discontinuously and co-compactly on the Cayley graph of G/N.

□

In geometric group theory we 'identify' groups which differ by a 'finite amount' as in the corollary above.

We give now some examples of algebraic properties that are preserved by quasi-isometries.

Exercise 6.2. Let $G =< S|R >$ be a finitely presented group and let H be a finitely generated group quasi-isometric to G. Then H is finitely presented.

Definition 6.6. If $G =< S >$ is a finitely generated group we define the growth function of G to be

$$vol_{S,G}(r) = |B(r)|$$

where $B(r)$ is the ball of radius r in (G, d_S) centered at e.

We define an equivalence relation on functions $f : \mathbb{R}^+ \to \mathbb{R}^+$. We say that $f \prec g$ if there are $A, B, C > 0$ such that for all $r \in \mathbb{R}^+$ we have $f(r) \leq Ag(Br) + C$. We note that $\prec$ is a partial order.

We say that $f \sim g$ if $f \prec g$ and $g \prec f$. $\sim$ is clearly an equivalence relation.

Exercise 6.3. Show that if $G_1 =< S >, G_2 =< S' >$ are finitely generated quasi-isometric groups then $vol_{S,G_1} \sim vol_{S',G_2}$. Deduce that the growth function of a group does not depend (up to equivalence) on the generating set that we pick.

Usually one considers this function up to equivalence, and denotes it by $vol_G(r)$.

Theorem 6.2. *(Gromov) A finitely generated group G has a nilpotent subgroup of finite index if and only if $vol_G(r) \prec r^n$ for some $n \in \mathbb{N}$.*

It follows from this theorem that if G is quasi-isometric to a finitely generated nilpotent group then G has a nilpotent subgroup of finite index.

Definition 6.7. (ends) Let Γ be a locally finite graph. If $K \subset \Gamma$ is compact we define $c(K)$ to be the number of unbounded connected components of $\Gamma - K$. We define then the *number of ends* of Γ to be

$$e(\Gamma) = \sup\{c(K) : K \subset \Gamma, \ \ compact\}$$

We remark that we obtain an equivalent definition if, instead of compact sets K, we consider finite sets of vertices of Γ. Clearly finite graphs have 0 ends.

For a finitely generated group G we define the number of ends, $e(G)$, of G to be the number of ends of the Cayley graph of G.

Exercise 6.4. Show that two quasi-isometric locally finite graphs have the same number of ends. Deduce that the number of ends of a finitely generated group is well defined (ie it does not depend on the Cayley graph that we pick).

Exercise 6.5. Show that a finitely generated group has 0,1,2 or ∞ ends.

For example $\mathbb{Z}^2$ has 1 end, $\mathbb{Z}$ has 2 ends while $\mathbb{F}_2$ has ∞ ends.

It turns out that the number of ends of the Cayley graph of a group tells us whether the group splits over a finite group:

Theorem 6.3. *(Stallings) A finitely generated group G splits over a finite group if and only if G has more than 1 end.*

It is easy to see (exercise) that if a f.g. group G splits over a finite group then $e(G) > 1$. So the interesting direction of the theorem is: if $e(G) > 1$ then G splits over a finite group.

Stallings theorem combined with Dunwoody's accessibility theorem implies that if a finitely generated group G is quasi-isometric to a free group F then it has a finite index subgroup which is free.

We treat now the easier case of groups quasi-isometric to $\mathbb{Z}$.

Proposition 6.1. *Let G be a finitely generated 2-ended group. Then G has a finite index subgroup isomorphic to $\mathbb{Z}$.*

Proof. Let Γ be the Cayley graph of G. We consider a compact connected set K containing e such that $\Gamma - K$ has 2 unbounded connected components C, D.

We claim that there is some $a \in G$ such that aC is properly contained in C.

Indeed pick g such that gK is contained in C. Then $D \cup K$ is an unbounded connected set of $\Gamma - gK$ that does not intersect gK. So it is properly contained either in gC or in gD. If $D \subset gD$ rename D to C and set $a = g^{-1}$. Otherwise $D \cup K \subset gC$, so $K \subset gC$ and $gD \cup gK$ is an unbounded connected set of $\Gamma - K$ so it is contained in either C or D. But if it is contained in D then it is contained in gC which is absurd. So $gD \subset C$.

Pick now h such that hK is contained in gD. If hD is properly contained in gD then $g^{-1}hD$ is properly contained in D. Set $a = g^{-1}h$ and rename D to C. Otherwise hC is properly contained in gD, hence it is properly contained in C.

We remark that $aC \subset C$ and $aC \neq C$. So $a^2C \subset aC \subset C$. Inductively we have $a^nC \subset C$, $a^nC \neq C$. It follows that a is an element of infinite order.

We note now that $K \cap aK = \emptyset$. Let v be any vertex of Γ. We claim that v is either contained in a^nK or v is contained in a bounded component of $\Gamma - (a^nK \cup a^{n+1}K)$ for some n. If v is contained in a bounded component of $\Gamma - a^nK$ for some n then the claim is proven. Otherwise we may assume that $v \in C$. Since $aC \subset C$ and $d(a^nK, e) \to \infty$ for some n, $d(a^nK, e) > d(e, v)$. It follows that v does not lie in a^nC as it can be joined to e by a path that does not intersect a^nK. Let n be maximum such that $v \in a^nC$. Then v lies in a connected component of $\Gamma - (a^nK \cup a^{n+1}K)$. If this component is unbounded then it is contained either in $a^{n+1}C$ or in a^nD but this is impossible, so v lies in a bounded connected component of $\Gamma - (a^nK \cup a^{n+1}K)$ and the claim is proven.

It follows that

$$\{a^n : n \in \mathbb{Z}\}$$

is a net in Γ. So $< a >$ is a finite index subgroup of G. □

Corollary 6.2. *Let G be a finitely generated group quasi-isometric to $\mathbb{Z}$. Then G has a finite index subgroup isomorphic to $\mathbb{Z}$.*

6.2 Hyperbolic Spaces

If X is a geodesic metric space, a geodesic triangle $[x, y, z]$ in X is a union of three geodesic paths $[x, y] \cup [y, z] \cup [x, z]$ where $x, y, z \in X$.

Definition 6.8. Let $\delta \geq 0$. We say that a geodesic triangle in a geodesic metric space is *δ-slim* if each side is contained in the δ-neighborhood of the two other sides. We say that a geodesic metric space X is *hyperbolic* if there is some $\delta \geq 0$ so that all geodesic triangles in X are δ-slim.

Examples. 1. Trees are hyperbolic spaces (in fact 0-hyperbolic).

2. Finite graphs are hyperbolic spaces.
3. $\mathbb{R}^2$ with the usual Euclidean metric is not hyperbolic.
4. It turns out that $\mathbb{H}^2$, the hyperbolic plane, is hyperbolic.

There are several equivalent formulations of hyperbolicity. We give one more now and we will discuss some other reformulations later in the course.

If $\Delta = [x, y, z]$ is a triangle then there is a metric tree (a 'tripod' if Δ is not degenerate) T_Δ with 3 points x', y', z' (the endpoints when T_Δ is not a segment) such that there is an onto map $f_\Delta : \Delta \to T_\Delta$ which restricts to an isometry from each side $[x, y], [y, z], [x, z]$ to the corresponding segments $[x', y'], [y', z'], [x', z']$. We denote by c_Δ the point $[x', y'] \cap [y', z'] \cap [x', z']$ of T_Δ.

Definition 6.9. Let $\delta \geq 0$. We say that a geodesic triangle $\Delta = [x, y, z]$ in a geodesic metric space is *δ-thin* if for every $t \in T_\Delta = [x', y', z']$, $diam(f_\Delta^{-1}(t)) \leq \delta$.

Theorem 6.4. *Let X be a geodesic metric space. The following are equivalent:*

1. *There is a $\delta \geq 0$ such that all geodesic triangles in X are δ-slim.*
2. *There is a $\delta' \geq 0$ such that all geodesic triangles in X are δ'-thin.*

Proof. Clearly 2 implies 1. Indeed one can simply take $\delta = \delta'$.

We show now that 1 implies 2. We will show that we may take $\delta' = 4\delta$.

Let $\Delta = [x, y, z]$ be a geodesic triangle and let $f_\Delta : \Delta \to T_\Delta$ the map defined above to a tripod. Let $f^{-1}(c_\Delta) = \{c_x, c_y, c_z\}$ where

$$c_x \in [y, z],\ c_y \in [x, z],\ c_z \in [x, y]$$

Let $a \in [x, c_z]$ and let a' in $[x, c_y]$ such that $d(x, a') = d(x, a)$. By symmetry it is enough to show that $d(a, a') \leq 4\delta$.

We have that

$$d(a, a_1) \leq \delta$$

for some

$$a_1 \in [x, z] \cup [y, z]$$

We distinguish two cases:

Case 1. $a_1 \in [x, z]$. Then

$$d(x, a') + \delta \geq d(x, a) + d(a, a_1) \geq d(x, a_1) \geq d(x, a) - d(a, a_1) \geq d(x, a') - \delta$$

by the triangle inequality. It follows that

$$d(a, a') \leq \delta + d(a_1, a') \leq 2\delta$$

Case 2. $a_1 \in [y, z]$. We claim that $d(a, c_x) \leq 2\delta$ in this case. Indeed if $a_1 \in [c_x, y]$ by the triangle inequality

$$d(a, y) \leq d(y, a_1) + \delta \implies d(y, a_1) \geq d(y, c_x) - \delta \implies d(a_1, c_x) \leq \delta$$

so $d(a, c_x) \leq 2\delta$.

If $a_1 \in [c_x, z]$ then again by the triangle inequality:

$$d(x, z) \leq d(x, a) + \delta + d(a_1, z) \implies d(x, z) \leq d(x, c_z) + \delta + d(a_1, z)$$

Since $d(x, z) = d(z, c_y) + d(x, c_z)$ and $d(z, c_y) = d(z, c_x)$ we obtain:

$$d(z, c_x) \leq d(a_1, z) + \delta$$

so $d(a_1, c_x) \leq \delta$ and $d(a, c_x) \leq 2\delta$. By symmetry, either, as in case 1, $d(a', a) \leq 2\delta$ or $d(a', c_x) \leq 2\delta$. It follows that

$$d(a, a') \leq 4\delta$$

□

Definition 6.10. Let X be a geodesic metric space. We say that X is δ-hyperbolic if all geodesic triangles in X are δ-thin.

Lemma 6.1. *Let X be a δ-hyperbolic geodesic metric space. Let $x_0, x_1, ..., x_n \in X$ and let $p \in [x_0, x_n]$. Then*

$$d(p, [x_0, x_1] \cup [x_1, x_2] ... \cup [x_{n-1}, x_n]) \leq (log_2(n) + 1)\delta$$

Proof. Let's say that $2^{k-1} < n \leq 2^k$ for $k \in \mathbb{N}$. It suffices to prove that

$$d(p, [x_0, x_1] \cup [x_1, x_2] ... \cup [x_{n-1}, x_n]) \leq k\delta.$$

We argue by induction on k. This is clearly true if $k = 1$ (ie. $n = 2$). For $k > 1$, pick $m = 2^{k-1}$. Then there is some $p_1 \in [x_0, x_m] \cup [x_m, x_n]$ with $d(p, p_1) \leq \delta$. By the inductive hypothesis

$$d(p_1, [x_0, x_1] \cup [x_1, x_2] ... \cup [x_{n-1}, x_n]) \leq (k-1)\delta$$

and the result follows.

□

6.3 Quasi-geodesics in hyperbolic spaces

Proposition 6.2 (Morse Lemma)**.** *Let X be a δ-hyperbolic metric space. There exist constants $L = L(\lambda, \mu), M = M(\lambda, \mu)$ such that if $x, y \in X$, $\alpha : I \to X$ is a (λ, μ)-quasi-geodesic with endpoints x, y and $\gamma = [x, y]$ then*

$$\gamma \subset N_L(\alpha), \ \alpha \subset N_M(\gamma)$$

Proof. According to Exercise 6.1, without loss of generality we can assume that $\alpha : I \to X$ is a (continuous) path of endpoints x, y such that for all $t, s \in I$,

$$length\,(\alpha([t, s])) \leq \lambda d(\alpha(t), \alpha(s)) + \mu$$

.

We show first the existence of L. Let $a \in \gamma$ such that $d(a, \alpha) = D$ is maximum. Let $a_1 \neq a_2 \in \gamma$ with

$$d(a, a_1) = d(a, a_2) = D$$

and let $\alpha(t), \alpha(s)$ points in α realizing $d(a_1, \alpha), d(a_2, \alpha)$, respectively. We consider the path

$$\beta = [a_1, \alpha(t)] \cup \alpha([t, s]) \cup [a_2, \alpha(s)]$$

Clearly $d(a, \beta) \geq D/2$.

We pick points $x_1 = \alpha(t), x_2, ..., x_{n-1} = \alpha(s)$ such that $d(x_i, x_{i+1}) = 1$ for $i = 1, ..., n-3$ and $d(x_{n-2}, x_{n-1}) \leq 1$. By lemma 6.1

$$d(a, [a_1, \alpha(t)] \cup [x_1, x_2] \cup ... \cup [x_{n-2}, x_{n-1}] \cup [a_2, \alpha(s)]) \leq (log_2(n) + 1)\delta$$

and

$$(log_2(n) + 1)\delta \geq \frac{D}{2} - 1 \Rightarrow (2n)^\delta \geq 2^{\frac{D}{2}-1}$$

Since $n - 3 \leq length\,(\alpha([t, s]))$ and $d(\alpha(t), \alpha(s)) \leq 4D$ we have $length\,(\alpha([t, s])) \leq 4D\lambda + \mu$ and we obtain:

$$(8D\lambda + 2\mu + 6)^\delta \geq 2^{\frac{D}{2}-1}$$

which gives a bound L for D that depends only on λ, μ (and δ).

We show now the existence of M. Let $x = \alpha(s)$. By an argument of connectedness there is some $y \in \gamma$ such that y is at distance at most L from $\alpha(s_1)$ and $\alpha(s_2)$ with $s_1 \leq s \leq s_2$. It follows that

$$length(\alpha([s_1, s_2]) \leq 2L\lambda + \mu,$$

therefore

$$d(x, \gamma) \leq L(2\lambda + 1) + \mu$$

so we may take $M = L(2\lambda + 1) + \mu$.

$\square$

Corollary 6.3. *Let X be a δ-hyperbolic metric space and let Y be a geodesic metric space quasi-isometric to X. Then Y is hyperbolic.*

Proof. Let Δ be a geodesic triangle in Y. If $f : Y \to X$ is a quasi-isometry $f(\Delta)$ is contained in a finite neighborhood of a (λ, μ) quasi-geodesic triangle Δ' in X, where λ, μ depend only on f. By proposition 6.2 Δ' is ϵ-thin for some $\epsilon = \epsilon(\lambda, \mu, \delta) \geq 0$. But then Δ is also δ'-thin for some δ' that depends only on δ and f. $\square$

6.4 Hyperbolic Groups

Definition 6.11. Let $G = \langle S \rangle$ where S is finite. We say that G is *hyperbolic* if the Cayley graph $\Gamma = \Gamma(S, G)$ is a hyperbolic metric space.

We say that G is *δ-hyperbolic* if all geodesic triangles in Γ are δ-thin.

Remark 6.1. By corollary 6.3 if $G = \langle S_1 \rangle = \langle S_2 \rangle$ with S_1, S_2 finite then $\Gamma(S_1, G)$ is hyperbolic if and only if $\Gamma(S_2, G)$ is hyperbolic, so the definition above does not depend on the generating set S.

We note that if a group G is not finitely generated then for $S = G$, $\Gamma(S, G)$ is bounded, hence hyperbolic. So one can not extend in any reasonable way the definition of hyperbolicity to groups that are not finitely generated.

Examples. 1. Finitely generated free (or virtually free) groups are hyperbolic.

2. Groups acting discretely and co-compactly on $\mathbb{H}^n$ are hyperbolic.

3. $\mathbb{Z}^2$ is *not* hyperbolic.

4. A finite presentation $\langle S|R\rangle$ is said to satisfy condition $C'(\frac{1}{7})$ if for any two cyclic permutations r_1, r_2 of words in $R \cup R^{-1}$ any common initial subword w of r_1, r_2 has length $|w| \leq \frac{1}{7}\min\{|r_1|, |r_2|\}$. It can be shown that $C'(\frac{1}{7})$-groups are hyperbolic. As an example the group

$$G = \langle a, b, c, d | abcdbadc\rangle$$

satisfies the $C'(\frac{1}{7})$ condition, so it is hyperbolic.

5. A theorem of Gromov-Olshanskii shows that 'statistically most groups are hyperbolic': Given $p, q \in \mathbb{N}$ consider all presentations of the form

$$\langle a_1, ..., a_p | r_1, ..., r_q\rangle$$

where the r_i's are cyclically reduced words of the a_j's. Let's denote by $N(t, \lambda t)$ (where $\lambda > 1$) all presentations of this type such that for all i,

$$t \leq |r_i| \leq \lambda t$$

We denote N_h the presentations of hyperbolic groups among those. Then

$$\lim_{t\to\infty} \frac{N_h}{N(t, \lambda t)} = 1$$

Definition 6.12. A *Dehn presentation* of a group G is a finite presentation $\langle S|R\rangle$ such that every reduced word $w \in F(S)$ which is equal to the identity in G contains more than half of a word in R.

Remark 6.2. If $\langle S|R\rangle$ is a Dehn presentation then the word problem for $\langle S|R\rangle$ is solvable. Indeed if w is a word we check if it contains more than half of a relation in R. If not then $w \neq 1$. Otherwise $w = w_1 u w_2$ for some $uv \in R$ with $|v| < |u|$. Then $w = w_1 v^{-1} w_2$ so we replace w by $w_1 v^{-1} w_2$ and we repeat. Since the length decreases this procedure terminates in finitely many steps.

Theorem 6.5. *Let $G = \langle S \rangle$ be a hyperbolic group. Then G has a Dehn presentation. In particular G is finitely presented and the word problem for G is solvable.*

Proof. Assume that triangles in $\Gamma = \Gamma(S, G)$ are δ-thin for $\delta \in \mathbb{N}$. We set

$$R = \{w \in F(S) : |w| \leq 10\delta,\ w \underset{G}{=} 1\}$$

We claim that $\langle S|R \rangle$ is a Dehn presentation for G. We will show that if $w \in F(S)$ is word such that $w \underset{G}{=} 1$ then w contains more than half of a word in R. We remark that this is trivially true if $|w| \leq 10\delta$. We see w as a closed path of length $n = |w|$ in the Cayley graph Γ, $w : [0, n] \to \Gamma$, $w(0) = w(n) = e$. If w contains a subword u of length $\leq 5\delta$ which is not geodesic then there is v with $|v| < |u|$ such that $uv \in R$, so w contains more than half of a relator and we are done. Otherwise let $t \in \{0, 1, 2, ..., n\}$ be such that $d(w(t), e)$ is maximum. We consider the triangles:

$$[e, w(t), w(t - 5\delta)], [e, w(t), w(t + 5\delta)]$$

Since these two triangles are δ-thin and $d(w(t), e) \geq d(w(t-5\delta), e)$, $d(w(t), e) \geq d(w(t+5\delta), e)$ we have that

$$d(w(t - 2\delta), w(t + 2\delta)) \leq 2\delta$$

so the subword of length 4δ, $[w(t - 2\delta), w(t + 2\delta)]$ is not geodesic. It follows that w contains more than half of a word in R.

□

Proposition 6.3. *Let G be a hyperbolic group. Then G has finitely many conjugacy classes of elements of finite order.*

Proof. Let $\langle S|R \rangle$ be a Dehn presentation of G. Let g be an element of finite order and let w be an element of the conjugacy class of g of minimal length. Then $w^n = 1$ so the word w^n contains more than half of a relation $r \in R$. We claim that

$$|w| \leq \frac{|r|}{2} + 2$$

Suppose not. We remark that w is cyclically reduced. We have then that $r = r_1 r_2$, with $|r_1| > |r_2|, |r_1| \leq \frac{|r|}{2} + 2$ and $w = utv$, $r_1 = vu$ for some words r_1, r_2, v, t, u where all the previous expressions are reduced. Then $u^{-1}wu = tvu = tr_1$ is in the conjugacy class of g. We have that $tr_1 = tr_2^{-1}$ and

$$|tr_2^{-1}| \leq |t| + |r_2| < |t| + |r_1| = |w|$$

which is a contradiction since w is an element of the conjugacy class of g of minimal length. We remark now that there are finitely many words w of length less than

$$\max\{\frac{|r|}{2} + 2 \,:\, r \in R\}$$

so there are finitely many conjugacy classes of elements of finite order. $\square$

We turn now our attention to the conjugacy problem. We recall that if $g \in G = \langle S \rangle$ we denote by $|g|$ the length of a shortest word on S representing g.

Lemma 6.2. *Let $G = \langle S|R \rangle$ be δ-hyperbolic (so triangles in $\Gamma(S, R)$ are δ-thin). If $g_1 \in G$ is conjugate to g_2 then there is some $x \in G$ such that $g_1 = xg_2x^{-1}$ and*

$$|x| \leq (2|S|)^{2\delta+|g_1|} + |g_1| + |g_2|$$

Proof. Let x be a word of minimal length such that $g_1 = xg_2x^{-1}$. Let's say that $x = x_1...x_n$ with $x_i \in S \cup S^{-1}$. We have then

$$|(x_1...x_i)^{-1}g_1(x_1...x_i)| \leq 2\delta + |g_1|$$

for all i with $|g_1| \leq i \leq n - |g_2|$. If

$$|x| \geq (2|S|)^{2\delta+|g_1|} + |g_1| + |g_2| + 1$$

then there are $i < j$ such that

$$(x_1...x_i)^{-1}g_1(x_1...x_i) = (x_1...x_j)^{-1}g_1(x_1...x_j)$$

so

$$(x_1...x_ix_{j+1}...x_n)^{-1}g_1(x_1...x_ix_{j+1}...x_n) = g_2$$

which contradicts the minimality of x.

$\square$

Corollary 6.4. *The conjugacy problem is solvable for hyperbolic groups.*

Proof. Indeed given $g_1, g_2 \in G$ it suffices to check whether $g_2 = xg_1x^{-1}$ for all x with

$$|x| \leq (2|S|)^{2\delta+|g_1|} + |g_1| + |g_2|$$

$\square$

Lemma 6.3. *Let $G = \langle S \rangle$ be δ-hyperbolic for some $\delta \in \mathbb{N}$, $\delta \geq 1$. Assume that for some $g \in G$ with $|g| > 4\delta$ we have that $|g^2| \leq 2|g| - 2\delta$. Then there is some $h \in G$ conjugate to g with $|h| < |g|$.*

Proof. Consider the triangle $[1, g, g^2]$ in $\Gamma(S, G)$. By δ-thinness of this triangle we have that there are $u, s, v \in G$ such that $g = usv$ (where usv is a geodesic word), $|u| = |v| = \delta$ and $|vu| \leq \delta$. If we set $t = vu$ we have that

$$g = usv = ustu^{-1}$$

and $|st| < |g|$.

□

Lemma 6.4. *Let $G = \langle S \rangle$ be δ-hyperbolic for some $\delta \in \mathbb{N}$, $\delta \geq 1$. Assume that for some $g \in G, x \in \Gamma(S, G)$ with $d(x, gx) > 100\delta$ we have that $d(x, g^2x) \geq 2d(x, gx) - 12\delta$. Then*

$$d(x, g^n x) \geq nd(x, gx) - 16n\delta$$

for all $n \in \mathbb{N}$.

Proof. It suffices to show that for all n

$$d(x, g^n x) \geq d(x, g^{n-1}x) + d(x, gx) - 16\delta$$

Clearly this holds for $n = 1, 2$. We argue by induction. Assume that it is true for all $k \leq n$. We consider the triangles $[x, g^n x, g^{n+1}x]$, $[x, g^{n-1}x, g^n x]$. Assume that

$$d(x, g^{n+1}x) < d(x, g^n x) + d(x, gx) - 16\delta$$

By δ-thinness of $[x, g^n x, g^{n+1}x]$ there are vertices u_1, u_2 on the geodesics $[g^n x, g^{n+1}x], [x, g^n x]$ respectively, such that

$$d(u_1, g^n x) = d(u_2, g^n x) = 8\delta, \; d(u_1, u_2) \leq \delta$$

Similarly by δ-thinness of $[x, g^{n-1}x, g^n x]$ there is a vertex $u_3 \in [g^{n-1}x, g^n x]$ such that $d(u_3, g^n x) = 8\delta$ and $d(u_2, u_3) \leq \delta$. We have then

$$d(x, g^2 x) = d(g^{n-1}x, g^{n+1}x) \leq d(g^{n-1}x, u_3) + d(u_1, u_3) + d(u_1, g^{n+1}x) = 2d(x, gx) - 14\delta$$

which is a contradiction.

□

Proposition 6.4. *Let $G = \langle S \rangle$ be δ-hyperbolic for some $\delta \in \mathbb{N}$, $\delta \geq 1$. Assume that g is an element of infinite order. Then there are constants $c > 0, d \geq 0$ such that*

$$d(1, g^n) \geq cn - d$$

for all $n \in \mathbb{N}$.

Proof. It is clear that we may replace g by a power, this will only affect the constants c, d in the statement. Further it is enough to show that for some $x \in \Gamma(S, G)$ there are constants c', d' so that

$$d(x, g^n x) \geq c'n - d'$$

for all n. Indeed by the triangle inequality

$$d(1, g^n) \geq d(x, g^n x) - d(1, x) - d(g^n, g^n x)$$

so

$$d(1, g^n) \geq d(x, g^n x) - 2d(1, x)$$

as $d(1, x) = d(g^n, g^n x)$.

In what follows we pick $n \gg k \gg 0, k, n \in \mathbb{N}$. It will be clear from the proof how k, n are chosen. We consider the geodesic $[1, g^n]$. Let m be a vertex on this geodesic at distance ≤ 1 from its midpoint. Let's say that there are R vertices in the ball $B(m, 100\delta)$. Then we we may pick $k \leq R + 1$ so that

$$d(m, g^k m) \geq 100\delta.$$

Let

$$M = max\{d(1, g^i) : 1 \leq i \leq R + 1\}$$

and let n be such that $d(1, g^n) > 10M$.

Now by thinness of the quadrilateral

$$[1, g^n, g^{k+n}, g^k]$$

we have that

$$d(g^k m, [1, g^n]) \leq 2\delta$$

In particular there is a vertex y on $[1, g^n]$ such that $d(y, g^k m) \leq 2\delta$. Then $g^k[m, y]$ is contained in the geodesic $[g^k, g^{k+n}]$ and there is some $z \in [1, g^n]$ such that $d(z, g^k y) \leq 2\delta$. It follows that

$$d(m, g^{2k} m) \geq d(m, z) - 4\delta$$

However $d(m, z) = d(m, y) + d(y, z)$ and by the triangle inequality

$$d(m, y) \geq d(m, g^k m) - 2\delta$$

and

$$d(y, z) \geq d(g^k m, g^k y) - 4\delta = d(m, y) - 4\delta \geq d(m, g^k m) - 6\delta$$

so

$$d(m, g^{2k} m) \geq 2d(m, g^k m) - 12\delta.$$

The assertion now follows by applying lemma 6.4 to g^k and m.

□

It follows from this proposition that if α is a geodesic from 1 to g then

$$\bigcup_n g^n \alpha$$

is a quasi-geodesic.

Proposition 6.5. *Let $G = \langle S \rangle$ be δ-hyperbolic and let $g \in G$ be an element of infinite order. Let $C(g)$ be the centralizer of g. Then the quotient $C(g)/\langle g \rangle$ is finite.*

Proof. Let $L > 0$ be such that for any $n \in \mathbb{N}$ the geodesic $[1, g^n]$ is contained in the L-neighborhood of $\{1, g, ..., g^n\}$. Let $s \in C(g)$ and $m \in \mathbb{N}$ such that

$$|g^m| \geq 2|s| + 2\delta$$

We consider the quadrilateral $[1, g^m, sg^m, s]$. By δ-thinness there is some vertex $p \in [1, g^m]$ such that

$$d(p, [s, sg^m]) \leq 2\delta$$

It follows that there are g^i, g^j such that

$$d(g^i, g^j s) \leq 2L + 2\delta$$

so

$$d(g^{i-j}, s) \leq 2L + 2\delta$$

It follows that $s = g^{i-j} u$ with $|u| \leq 2L + 2\delta$. Therefore every coset $s\langle g \rangle$ has a representative which has word length $\leq 2L + 2\delta$. Hence the quotient $C(g)/\langle g \rangle$ is finite.

□

Corollary 6.5. *If G is hyperbolic then G has no subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

6.5 More results and open problems

There is a number of results on hyperbolic groups that we were not able to present in this short introduction. We give a list of some results hoping that this will give a better perspective on the subject. Some of the results below can be proven by the techniques that we have already presented while others are quite deep requiring a quite different approach.

Theorem 6.6. *Let G be a hyperbolic group which is not finite or virtually $\mathbb{Z}$. Then G contains a free subgroup of rank 2.*

Theorem 6.7. *Let G be a hyperbolic group and let $g_1, ..., g_n \in G$. Then there is some $N > 0$ such that the group $\langle g_1^N, ..., g_n^N \rangle$ is free.*

Theorem 6.8. *(Gromov-Delzant) Let G be a hyperbolic group and let H be a fixed one-ended group. Then G contains at most finitely many conjugacy classes of subgroups isomorphic to H.*

Theorem 6.9. *(Sela-Guirardel-Dahmani) The isomorphism problem is solvable for hyperbolic groups.*

Theorem 6.10. *(Sela) Torsion free hyperbolic groups are Hopf.*

There is a number of open questions about hyperbolic groups:

1. Are hyperbolic groups residually finite?

2. Let G be hyperbolic. Does G have a torsion free subgroup of finite index?

3. Gromov conjectures that if G is torsion free hyperbolic then G has finitely many torsion free finite extensions.