C3.11 Riemannian Geometry

Sheet 3 — HT24

Solutions

This problem sheet is based on Sections 4–6 of the lecture notes. This version contains the solutions to Sections A and C.

Section A

1. Let $(\mathcal{S}^n, g)$ be the round n-sphere and let h be the product metric on $\mathcal{S}^n \times \mathcal{S}^n$.

   Show that $(\mathcal{S}^n \times \mathcal{S}^n, h)$ is Einstein with non-negative sectional curvature.

   Solution: We recall from Sheet 1 that, if ∇ is the Levi-Civita connection of the product metric on $M_1 \times M_2$ and ∇_1, ∇_2 are the Levi-Civita connections on $(M_1, g_1), (M_2, g_2)$, then

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) = ((\nabla_1)_{X_1} Y_1, (\nabla_2)_{X_2} Y_2)$$

   for all vector fields X_1, Y_1 on M_1 and X_2, Y_2 on M_2. Notice also that

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

   Therefore, if R_1, R_2 are the Riemann curvatures of g_1, g_2,

$$R((X_1, X_2), (Y_1, Y_2))(Z_1, Z_2) = (R_1(X_1, Y_1)Z_1, R_2(X_2, Y_2)Z_2)$$

   for all vector fields (or tangent vectors) X_1, Y_1, Z_1 on M_1 and vectors fields (or tangent vectors) X_2, Y_2, Z_2 on M_2.

   Since the product metric h satisfies

$$h((X_1, X_2), (Y_1, Y_2)) = g_1(X_1, Y_1) + g_2(X_2, Y_2)$$

   we deduce that

$$R((X_1, X_2), (Y_1, Y_2), (Z_1, Z_2), (W_1, W_2) = R_1(X_1, Y_1, Z_1, W_1) + R_2(X_2, Y_2, Z_2, W_2).$$

   Hence,

$$\mathrm{Ric}((X_1, X_2), (Y_1, Y_2)) = \mathrm{Ric}_1(X_1, Y_1) + \mathrm{Ric}_2(X_2, Y_2)$$

   (since we can construct an orthonormal basis for $T_{(p_1,p_2)} M_1 \times M_2$ from a union of orthonormal bases for $T_{p_1} M_1$ and $T_{p_2} M_2$).

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In our case, $M_1 = M_2 = \mathcal{S}^n$ with the round metric g, so $\mathrm{Ric}_j = (n-1)g_j$ and therefore

$$\begin{aligned}\mathrm{Ric}((X_1,X_2),(Y_1,Y_2)) &= (n-1)g_1(X_1,Y_1) + (n-1)g_2(X_2,Y_2)\\ &= (n-1)g((X_1,X_2),(Y_1,Y_2)).\end{aligned}$$

Hence g is Einstein.

Moreover, if X, Y are tangent vectors on $(\mathcal{S}^n, g)$, $R_1(X,Y,Y,X) = R_2(X,Y,Y,X) \geq 0$ since g has constant sectional curvature 1 and therefore

$$R((X_1,X_2),(Y_1,Y_2),(Y_1,Y_2),(X_1,X_2) \geq 0.$$

We deduce that $K \geq 0$ on $(\mathcal{S}^n \times \mathcal{S}^n, h)$.

[Notice that $(\mathcal{S}^n \times \mathcal{S}^n, h)$ is *never* positively curved:

$$R((X_1,0),(0,Y_2),(0,Y_2),(X_1,0)) = 0$$

so $K((X_1,0),(0,Y_2)) = 0$. In fact, the *Hopf conjecture* asserts that $\mathcal{S}^2 \times \mathcal{S}^2$ *does not admit* a metric with positive sectional curvature.]

2. (a) Show that the induced metric on an oriented minimal hypersurface in $(\mathbb{R}^{n+1}, g_0)$ is flat if and only if the minimal hypersurface is totally geodesic.

   (b) Let
   $$M = \{(z_1,z_2) \in \mathbb{C}^2 \,:\, |z_1| = |z_2| = \frac{1}{\sqrt{2}}\} \subseteq \mathcal{S}^3$$
   and let g be the induced metric on M from the round metric on $\mathcal{S}^3$.
   Show that (M,g) is flat and that M is a minimal hypersurface in $\mathcal{S}^3$ which is not totally geodesic.

Solution:

(a) Let M be an oriented minimal hypersurface in $(\mathbb{R}^{n+1}, g_0)$. Let $p \in M$ and let $\{E_1,\ldots,E_n\}$ denote the principal directions at p and let $\lambda_1,\ldots,\lambda_n$ be the associated principal curvatures. By the Gauss equation, we have that

$$\begin{aligned}0 &= K^{\mathbb{R}^{n+1}}(E_i,E_j)\\ &= K^M(E_i,E_j) + g(B(E_i,E_j),B(E_i,E_j)) - g(B(E_i,E_i),B(E_j,E_j)).\end{aligned}$$

Since E_i are principal directions, if ν is the Gauss map on M, we have that

$$g(B(E_i,E_j),\nu) = g(S_\nu E_i, E_j) = \lambda_i\delta_{ij}$$

and thus

$$B(E_i, E_j) = \lambda_i \delta_{ij} \nu.$$

Therefore,

$$K^M(E_i, E_j) = \lambda_i \lambda_j.$$

We deduce that M is flat if and only if $\lambda_i \lambda_j = 0$ for all i, j. Hence, M is flat if and only if all but at most one of the λ_i is zero. However, since M is minimal we have that $\sum_{i=1}^n \lambda_i = 0$, and therefore M is flat if and only if $\lambda_1 = \ldots = \lambda_n = 0$, which is the statement that $B = 0$.

(b) We define an immersion $f : \mathbb{R}^2 \to \mathbb{C}^2$ by

$$f(\theta_1, \theta_2) = \frac{1}{\sqrt{2}}(e^{i\theta_1}, e^{i\theta_2})$$

so that $f(\mathbb{R}^2) = M$. Identifying vector fields in $\mathbb{C}^2$ with vectors, we see that

$$X_1 = f_*(\partial_1) = \frac{i}{\sqrt{2}}(e^{i\theta_1}, 0)$$
$$X_2 = f_*(\partial_2) = \frac{i}{\sqrt{2}}(0, e^{i\theta_2}).$$

We deduce that, since the round metric on $\mathcal{S}^3$ is induced from the Euclidean metric g_0 on $\mathbb{C}^2$, we have that $f^*g = f^*g_0$. Now,

$$g_0(X_1, X_1) = \frac{1}{2} = g_0(X_2, X_2)$$
$$g_0(X_1, X_2) = 0$$

and hence

$$f^*g = \frac{1}{2}(\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2).$$

Since this is a rescaling of the Euclidean metric, which is flat, we deduce that f^*g is flat, and hence that g is flat.

Recall by the Gauss equation we have

$$\begin{aligned} R^{\mathcal{S}^3}(X_1, X_2, X_2, X_1) &= R^M(X_1, X_2, X_2, X_1) \\ &\quad + g(B(X_1, X_2), B(X_1, X_2)) - g(B(X_1, X_1), B(X_2, X_2)) \end{aligned}$$

where B is the second fundamental form of M in $\mathcal{S}^3$. Since $\mathcal{S}^3$ has constant curvature 1 and g is flat, and $g(X_1, X_1)g(X_2, X_2) - g(X_1, X_2)^2 = \frac{1}{4}$, we have that

$$\frac{1}{4} = g(B(X_1, X_2), B(X_1, X_2)) - g(B(X_1, X_1), B(X_2, X_2))$$

and so M is not totally geodesic.

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Since g is a multiple of the Euclidean metric, we have that

$$\nabla^M_{X_i} X_j = 0.$$

Hence, if we let $E_1 = \sqrt{2}X_1$ and $E_2 = \sqrt{2}X_2$, which are orthonormal, then the mean curvature of M in $\mathcal{S}^3$ is

$$B(E_1, E_1) + B(E_2, E_2) = \nabla^{\mathcal{S}^3}_{E_1} E_1 + \nabla^{\mathcal{S}^3}_{E_2} E_2.$$

We now compute $\nabla^{\mathbb{C}^2}_{X_i} X_i$ as:

$$\nabla^{\mathbb{C}^2}_{X_1} X_1 = -\frac{1}{\sqrt{2}}(e^{i\theta_1}, 0),$$
$$\nabla^{\mathbb{C}^2}_{X_2} X_2 = -\frac{1}{\sqrt{2}}(0, e^{i\theta_2}).$$

Hence,

$$\nabla^{\mathbb{C}^2}_{E_1} E_1 + \nabla^{\mathbb{C}^2}_{E_2} E_2 = -\sqrt{2}(e^{i\theta_1}, e^{i\theta_2}).$$

Since this vector field is normal to $\mathcal{S}^3$, and the round metric on $\mathcal{S}^3$ is induced from the Euclidean metric on $\mathbb{C}^2$, we deduce that

$$H = \nabla^{\mathcal{S}^3}_{E_1} E_1 + \nabla^{\mathcal{S}^3}_{E_2} E_2 = 0.$$

Thus, M is minimal as claimed.

[The submanifold M is called the *Clifford torus* in $\mathcal{S}^3$. and is very important in geometry and topology, including links to symplectic geometry. There are many interesting open questions still regarding the Clifford torus, and is the subject of the *Lawson* and *Willmore* conjectures from the mid-20th century, both of which were only solved quite recently.]

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Section B

3. Let E_1, E_2, E_3 be vector fields on $\mathcal{S}^3$ such that $[E_i, E_j] = -2\epsilon_{ijk}E_k$. For $\lambda > 0$, let

$$X_1 = \lambda E_1, \quad X_2 = E_2, \quad X_3 = E_3$$

and define a Riemannian metric g on $\mathcal{S}^3$ by the condition that

$$g(X_i, X_j) = \delta_{ij}$$

(a) Show that $(\mathcal{S}^3, g)$ is Einstein if and only if $\lambda = 1$.

(b) Find a necessary and sufficient condition on λ so that the scalar curvature of $(\mathcal{S}^3, g)$ is zero.

4. Let M be $\mathrm{SO}(n)$, $\mathrm{O}(n)$, $\mathrm{SU}(m)$ or $\mathrm{U}(m)$ and let g be the bi-invariant metric on M given by

$$g_A(B, C) = -\operatorname{tr}(A^{-1}BA^{-1}C)$$

for all $A \in M$ and $B, C \in T_AM$. Let $L_A : M \to M$ denote left-multiplication by A and let

$$\mathcal{X} = \{\text{vector fields } X \text{ on } M \,:\, (L_A)_*X = X \,\forall A \in M\}.$$

(a) Show that, for all $X, Y \in \mathcal{X}$,

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

[*You may assume that* $[X, Y](I)$ *is the matrix commutator of* $X(I)$ *and* $Y(I)$*, where* I *is the identity matrix.*]

(b) Show that the sectional curvatures of (M, g) are non-negative and that (M, g) is flat if and only if $n = 2$ or $m = 1$.

(c) Let $m > 1$ and define a submanifold D of $\mathrm{U}(m)$ by

$$D = \{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_m}) \,:\, \theta_1, \ldots, \theta_m \in \mathbb{R}\} \subseteq \mathrm{U}(m).$$

Show that D is a flat totally geodesic submanifold in $(\mathrm{U}(m), g)$.

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5. (a) Let $\gamma : [0, L] \to (M, g)$ be a geodesic and let $f : (-\epsilon, \epsilon) \times [0, L] \to M$ be a variation of γ so that the curve $\gamma_s : [0, L] \to (M, g)$ given by $\gamma_s(t) = f(s, t)$ is a geodesic for all $s \in (-\epsilon, \epsilon)$.

   Show that the variation field V_f of f is a Jacobi field along γ.

   (b) Let

$$\mathcal{H}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}$$

   and let g be the restriction of $h = \sum_{i=1}^{n} \mathrm{d}x_i^2 - \mathrm{d}x_{n+1}^2$ on $\mathbb{R}^{n+1}$ to $\mathcal{H}^n$. Given that the normalized geodesics γ in $(\mathcal{H}^n, g)$ with $\gamma(0) = x$ and $\gamma'(0) = X$ are given by

$$\gamma(t) = x \cosh t + X \sinh t,$$

   show that $(\mathcal{H}^n, g)$ has constant sectional curvature -1.

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Section C

6. Let $(\mathcal{S}^{2n+1}, g)$ be the round $(2n+1)$-sphere, view $\mathcal{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ and let $\pi : \mathcal{S}^{2n+1} \to \mathbb{CP}^n$ be the projection map. For $z \in \mathcal{S}^{2n+1}$ we have $E(z) = iz$ (identifying tangent vectors in $\mathbb{C}^n$ with $\mathbb{C}^n$), $\ker \mathrm{d}\pi_z = \mathrm{Span}\{E(z)\}$ and we let

$$H_z = \{X \in T_z\mathcal{S}^{2n+1} \ : \ g(X, E(z)) = 0\} \quad \text{and} \quad \Phi_z = \mathrm{d}\pi_z : H_z \to T_{\pi(z)}\mathbb{CP}^n.$$

The Fubini–Study metric h on $\mathbb{CP}^n$ is then given by

$$h_{\pi(z)}(X, Y) = g_z(\Phi_z^{-1}(X), \Phi_z^{-1}(Y)).$$

(a) For any vector field X on $\mathbb{CP}^n$ we define a vector field $\widehat{X}$ on $\mathcal{S}^{2n+1}$ by

$$\widehat{X}(z) = \Phi_z^{-1}\big(X(\pi(z))\big).$$

If $\widehat{\nabla}$ is the Levi-Civita connection of g and ∇ is the Levi-Civita connection of h, show that, for all vector fields X, Y on $\mathbb{CP}^n$

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y} = \widehat{\nabla_X Y} + \frac{1}{2}g([\widehat{X}, \widehat{Y}], E)E.$$

[Hint: Show that $[\widehat{X}, \widehat{Y}] - \widehat{[X, Y]}$ and $[\widehat{X}, E]$ are multiples of E.]

(b) Show that $\gamma : (-\epsilon, \epsilon) \to (\mathbb{CP}^n, h)$ is a geodesic with $\gamma(0) = \pi(z)$ if and only if $\gamma = \pi \circ \hat{\gamma}$ where $\hat{\gamma} : (-\epsilon, \epsilon) \to (\mathcal{S}^{2n+1}, g)$ is a geodesic with $\hat{\gamma}(0) = z$ and $\hat{\gamma}'(0) \in H_z$.

(c) Since $X \in H_z$ if and only if $iX \in H_z$, we can define $J = J_{\pi(z)} : T_{\pi(z)}\mathbb{CP}^n \to T_{\pi(z)}\mathbb{CP}^n$ by

$$J(X) = \mathrm{d}\pi_z(i\Phi_z^{-1}(X)),$$

which then extends to a map J from vector fields to vector fields on $\mathbb{CP}^n$. Let $X, Y \in T_{\pi(z)}\mathbb{CP}^n$ be orthogonal unit vectors and write $Y = \cos\alpha Z + \sin\alpha JX$ where Z is orthogonal to JX and unit length. Show that the sectional curvature K of $(\mathbb{CP}^n, h)$ satisfies

$$K(X, Y) = 1 + 3\sin^2\alpha.$$

[Hint: Let γ be a geodesic in $(\mathbb{CP}^n, h)$ with $\gamma(0) = \pi(z)$ and $\gamma'(0) = X$, and consider a variation $f(s, t)$ of γ so that $\gamma_s(t) = f(s, t)$ is geodesic for all s such that $\gamma_s(0) = \pi(z)$ and $\gamma_s'(0) = \cos sX + \sin sY$. You may want to consider the cases $\sin\alpha = 0$ and $\cos\alpha = 0$ first.]

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Solution:

(a) We see that

$$d\pi_z[\widehat{X},\widehat{Y}](z) = [d\pi_z\widehat{X}(z), d\pi_z\widehat{Y}(z)] = [X,Y](z),$$

by the relationship between the differential of smooth maps and the Lie bracket, and that

$$d\pi_z\widehat{[X,Y]}(z) = [X,Y](z)$$

by definition. Hence $[\widehat{X},\widehat{Y}] - \widehat{[X,Y]}$ lies in $\ker d\pi_z$ at all points $z \in \mathcal{S}^{2n+1}$ and thus must be a multiple of E.

We deduce from that, for all vector fields Z on $\mathbb{CP}^n$,

$$g([\widehat{X},\widehat{Y}],\widehat{Z}) = g(\widehat{[X,Y]},\widehat{Z}) = h([X,Y],Z).$$

Therefore, by the Koszul formula,

$$\begin{aligned} g(\widehat{\nabla}_{\widehat{X}}\widehat{Y},\widehat{Z}) &= \frac{1}{2}\Big(\widehat{X}\big(g(\widehat{Y},\widehat{Z})\big) + \widehat{Y}\big(g(\widehat{Z},\widehat{X})\big) - \widehat{Z}\big(g(\widehat{X},\widehat{Y})\big) \\ &\qquad - g(\widehat{X},[\widehat{Y},\widehat{Z}]) + g(\widehat{Y},[\widehat{Z},\widehat{X}]) + g(\widehat{Z},[\widehat{X},\widehat{Y}])\Big) \\ &= \frac{1}{2}\Big(X\big(h(Y,Z)\big) + Y\big(h(Z,X)\big) - Z\big(h(X,Y)\big) \\ &\qquad - h(X,[Y,Z]) + h(Y,[Z,X]) + h(Z,[X,Y])\Big) \\ &= h(\nabla_X Y, Z) = g(\widehat{\nabla_X Y},\widehat{Z}), \end{aligned}$$

since $\widehat{X}(g(\widehat{Y},\widehat{Z})) = X(h(Y,Z))$. We deduce that

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y} - \widehat{\nabla_X Y}$$

must be a multiple of E.

We then see that

$$d\pi_z[\widehat{X},E](z) = [X(\pi(z)), d\pi_z(E(z))] = 0$$

and hence $[\widehat{X},E]$ must be a multiple of E and thus $g([\widehat{X},E],\widehat{Y}) = 0$. In the Koszul formula we also see that $\widehat{X}(g(\hat{Y},E)) = 0$, $E(g(\widehat{X},\hat{Y})) = 0$, so

$$\begin{aligned} g(\widehat{\nabla}_{\widehat{X}}\widehat{Y},E) &= \frac{1}{2}\Big(\widehat{X}\big(g(\widehat{Y},E)\big) + \widehat{Y}\big(g(E,\widehat{X})\big) - E\big(g(\widehat{X},\widehat{Y})\big) \\ &\qquad - g(\widehat{X},[\widehat{Y},E]) + g(\widehat{Y},[E,\widehat{X}]) + g(E,[\widehat{X},\widehat{Y}])\Big) \\ &= g([\widehat{X},\widehat{Y}],E), \end{aligned}$$

which gives the result.

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(b) Let γ be a geodesic in $(\mathbb{CP}^n, h)$ with $\gamma(0) = \pi(z)$ and $\gamma'(0) = X \in T_{\pi(z)}\mathbb{CP}^n$. There exists a unique geodesic $\hat{\gamma}$ in $(\mathcal{S}^{2n+1}, g)$ with $\hat{\gamma}(0) = z$ and $\hat{\gamma}'(0) = \hat{X} = \Phi_z^{-1}X \in H_z$.

Since the flow ϕ_t^E of E is multiplication by e^{it} which is an isometry on $(\mathcal{S}^{2n+1}, g)$, E is a Killing field and hence, by the Killing equation,

$$g(\hat{\gamma}', \widehat{\nabla}_{\hat{\gamma}'}E) = 0.$$

Since $\hat{\gamma}$ is a geodesic,

$$\hat{\gamma}'(g(\hat{\gamma}', E)) = g(\widehat{\nabla}_{\hat{\gamma}'}\hat{\gamma}, E) + g(\hat{\gamma}', \widehat{\nabla}_{\hat{\gamma}'}E) = 0.$$

Therefore, as $\hat{\gamma}'(0) \in H_z$, $\hat{\gamma}'(s) \in H_z$ for all s.

If we let $\alpha = \pi \circ \hat{\gamma}$, then $\alpha(0) = \gamma(0)$, $\alpha'(0) = \gamma'(0)$ and $\widehat{\alpha'} = \hat{\gamma}'$ along $\hat{\gamma}$. We deduce from (a) that

$$0 = \widehat{\nabla}_{\hat{\gamma}'}\hat{\gamma}' = \widehat{\nabla}_{\widehat{\alpha'}}\widehat{\alpha'} = \widehat{\nabla_{\alpha'}\alpha'}$$

since the Lie bracket term from the formula vanishes. Hence, α is a geodesic and so, by uniqueness of geodesics, $\gamma = \alpha = \pi \circ \hat{\gamma}$.

We have also shown, with this argument, that if $\hat{\gamma}$ is a geodesic in $(\mathcal{S}^{2n+1}, g)$ with $\hat{\gamma}(0) = z$ and $\hat{\gamma}'(0) \in H_z$, then $\gamma = \pi \circ \hat{\gamma}$ is a geodesic in $(\mathbb{CP}^n, h)$ with $\gamma(0) = \pi(z)$.

(c) Let γ be a geodesic with $\gamma(0) = \pi(z)$ and $\gamma'(0) = X$. Let $\widehat{X} = \Phi_z^{-1}(X)$ and $\widehat{Y} = \Phi_z^{-1}(Y)$ in H_z. Consider the geodesics $\hat{\gamma}_s$ in $(\mathcal{S}^{2n+1}, g)$ given by

$$\hat{\gamma}_s(t) = z\cos t + (\cos s\widehat{X} + \sin s\widehat{Y})\sin t.$$

We can then define a variation f of γ by

$$f(s,t) = \pi(\hat{\gamma}_s(t)).$$

By (b) we have that $\gamma_s(t) = f(s,t) = \pi \circ \hat{\gamma}_s(t)$ is a geodesic in $(\mathbb{CP}^n, h)$ for all s. Therefore, by Question 5, we know that

$$V_f'' + R(V_f, \gamma')\gamma' = 0.$$

We see that

$$V_f(t) = \frac{\partial f}{\partial s}(0,t) = \mathrm{d}\pi_{\hat{\gamma}(t)}(\widehat{Y}\sin t).$$

Now here we have to be careful since $\hat{Y} \in H_z$ must that *does not mean* that $\hat{Y} \in H_{\hat{\gamma}(t)}$ for all t. In fact, we see that

$$\begin{aligned} g_{\hat{\gamma}(t)}(\widehat{Y}, E) &= g_{\hat{\gamma}(t)}(\cos\alpha\widehat{Z} + \sin\alpha i\widehat{X}, i\hat{\gamma}(t)) \\ &= g_{\hat{\gamma}(t)}(\cos\alpha\widehat{Z} + i\sin\alpha\widehat{X}, iz\cos t + i\widehat{X}\sin t) \\ &= \sin\alpha\sin t \end{aligned}$$

since $g(\widehat{Z}, iz) = 0$ as $\widehat{Z} \in H_z$, $g(i\widehat{X}, iz) = g(\widehat{X}, z) = 0$ as $\widehat{X} \in T_z\mathcal{S}^{2n+1}$, and $g(\widehat{Z}, i\widehat{X}) = g(Z, JX) = 0$. Hence,

$$\begin{aligned}\widehat{Y} - \sin\alpha\sin t i\hat{\gamma}(t) &= \cos\alpha\widehat{Z} + i\sin\alpha\widehat{X} - iz\sin\alpha\sin t\cos t - i\sin\alpha\sin^2 t\widehat{X} \\ &= \cos\alpha\widehat{Z} + i\sin\alpha\cos t(-z\sin t + \widehat{X}\cos t) \\ &= \cos\alpha\widehat{Z} + i\sin\alpha\cos t\hat{\gamma}'(t)\end{aligned}$$

lies in $\ker \mathrm{d}\pi_{\hat{\gamma}(t)}$ for all t.

Taking $\sin\alpha = 0$ and $\cos\alpha = 1$, we see that $\widehat{Y} = \widehat{Z}$ does in fact lie in $H_{\hat{\gamma}(t)}$ for all t and thus

$$V_f(t) = Z\sin t,$$

which satisfies $V_f'' = -Z\sin t = -V_f$. Therefore, as Z is unit,

$$R(Z\sin t, \gamma')\gamma' = -V_f'' = Z\sin t.$$

Dividing by $\sin t$ and letting $t \to 0$ gives

$$R(Z, X)X = Z.$$

Taking $\cos\alpha = 0$ and $\sin\alpha = 1$, we see that $i\sin t\cos t\hat{\gamma}'t$ is the component of $\widehat{Y}\sin t$ lying in $\ker \mathrm{d}\pi_{\hat{\gamma}(t)}$ and thus

$$V_f = \sin t\cos tJ\gamma'.$$

To compute V_f'' it is useful to show that $J\gamma'$ is parallel along γ. Now, by (a),

$$\widehat{\nabla}_{\hat{\gamma}'}\widehat{(J\gamma')} - \widehat{\nabla_{\gamma'}J\gamma')} = \widehat{\nabla}_{\hat{\gamma}'}(i\hat{\gamma}' - \widehat{\nabla_{\gamma'}J\gamma'}$$

is a multiple of E. We then can compute

$$\widehat{\nabla}_{\hat{\gamma}'}(i\hat{\gamma}') = \widehat{\nabla}_{\hat{\gamma}'}(-iz\sin t + i\widehat{X}\cos t) = -iz\cos t - i\widehat{X}\sin t = -i\hat{\gamma}(t),$$

which is a multiple of E, and hence $\widehat{\nabla_{\gamma'}J\gamma'}$ must be a multiple of E which is thus 0. Therefore $J\gamma'$ is parallel along γ and so

$$V_f'' = -4\sin t\cos tJ\gamma' = -4\sin t\cos tV_f.$$

Using the Jacobi equation again, we see that

$$R(\sin t\cos J\gamma', \gamma')\gamma' = -V_f'' = 4\sin t\cos tJ\gamma'$$

and thus

$$R(JX, X)X = 4JX.$$

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In general, by linearity, we then have

$$R(Y,X)X = \cos\alpha R(Z,X)X + \sin\alpha R(JX,X)X = \cos\alpha Z + 4\sin\alpha JX.$$

We deduce that, since X, Y are orthonormal and Z is orthogonal to JX,

$$\begin{aligned} K(X,Y) &= R(Y,X,X,Y) \\ &= h(\cos\alpha Z + 4\sin\alpha JX, \cos\alpha Z + \sin\alpha JX) \\ &= \cos^2\alpha + 4\sin^2\alpha = 1 + 3\sin^2\alpha \end{aligned}$$

as claimed.

[We have shown that $(\mathbb{CP}^n, h)$ has $1 \leq K \leq 4$. The *Sphere Theorem* states that if we have any simply connected (M, g) $1 < K \leq 4$ then (M, g) is homeomorphic to $\mathcal{S}^n$. (In fact, we now know that (M, g) must be *diffeomorphic* to $\mathcal{S}^n$, which is surprising given the existence of exotic spheres where are homeomorphic but not diffeomorphic to $\mathcal{S}^n$.) Therefore $(\mathbb{CP}^n, h)$ shows that the statement of the Sphere Theorem is sharp in even dimensions at least 4 since $\mathbb{CP}^n$ is simply connected but not homeomorphic to $\mathcal{S}^{2n}$ for n at least 2. The case of odd dimensions is still a subject of current research.]

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