C3.11 Riemannian Geometry

Sheet 4 — HT24

Solutions

This problem sheet is based on Sections 7–9 of the lecture notes. This version contains solutions to all questions.

Section A

1. Let $f : (M, g) \to (N, h)$ be a surjective local isometry between connected Riemannian manifolds.

   (a) Show that if (M, g) is complete then (N, h) is complete.

   (b) If (N, h) is complete, is (M, g) complete? Give a proof or a counterexample.

   Let $(\widetilde{M}, \tilde{g})$ be the universal cover of (M, g) with the covering metric.

   (c) Show that $(\widetilde{M}, \tilde{g})$ is complete if and only if (M, g) is complete.

   Solution:

   (a) Let $q \in N$. There exists $p \in M$ with $f(p) = q$ since f is surjective and open sets $V \ni p$ and $U \ni q$ such that $f : V \to U$ is an isometry.

   Let $\alpha : (-\epsilon, \epsilon) \to N$ be a geodesic in (N, h) through q contained in U. Let $\beta = f^{-1} \circ \alpha : (-\epsilon, \epsilon) \to V$. Since $f|_V$ is an isometry, β is a geodesic in (M, h) through p.

   Since (M, g) is complete, β is defined on $\mathbb{R}$, so $\beta : \mathbb{R} \to M$ is a geodesic through p. Thus $\gamma : \mathbb{R} \to N$ given by $\gamma = f \circ \beta$ is a curve in N through q which is a geodesic as f is a local isometry. Since $\gamma(t) = \alpha(t)$ for all $t \in (-\epsilon, \epsilon)$, we have $\gamma(0) = \alpha(0)$ and $\gamma'(0) = \alpha'(0)$, so uniqueness of geodesics implies that $\gamma = \alpha$ and hence (N, h) is complete.

   (b) It is not necessarily the case because we can take $\mathcal{S}^2 \setminus \{N\}$, where N is the North pole, which is now not complete, because geodesics that used to pass through N now are no longer defined on all of $\mathbb{R}$ (in particular, if we look at the exponential map from the South pole S, normalized geodesics γ are only defined for $t \in (-\pi, \pi)$). However the projection map $\pi : \mathcal{S}^2 \setminus \{N\} \to \mathbb{RP}^2$ is still a local isometry and it is still surjective because $\pi(S) = \pi(N)$ and $\mathbb{RP}^2$ is complete.

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(c) Since the covering map $\pi : (\widetilde{M}, \tilde{g}) \to (M, g)$ is a surjective local isometry, (a) shows that if $(\widetilde{M}, \tilde{g})$ is complete then (M, g) is complete.

Suppose now that (M, g) is complete. Let $p \in \widetilde{M}$ and let $\beta : (-\epsilon, \epsilon) \to (\widetilde{M}, \tilde{g})$ be a geodesic with $\beta(0) = p$. Then, making ϵ smaller if necessary so that the image of β is contained in an open set on which π restricts to be an isometry, we have that $\alpha = \pi \circ \beta : (-\epsilon, \epsilon) \to (M, g)$ is a geodesic with $\alpha(0) = \pi(p)$.

Since (M, g) is complete, α can be extend to all of $\mathbb{R}$ to give a geodesic $\alpha : \mathbb{R} \to (\tilde{M}, \tilde{g})$. Since π is a covering map, by the curve-lifting property there exists a curve $\gamma : \mathbb{R} \to (\widetilde{M}, \tilde{g})$ such that $\gamma(0) = p$ and $\pi \circ \gamma = \alpha$. Since π is a local isometry we have that γ must be a geodesic: explicitly, given any $t \in \mathbb{R}$ there exist open sets $U \ni \gamma(t)$ and $V \ni \alpha(t)$ such that $\pi : U \to V$ is an isometry and so $gamma|_U = \pi^{-1}|_V \circ \alpha|_V$ and thus is a geodesic on U and hence at $\gamma(t)$ for all t.

We then see that $\beta(0) = \gamma(0)$ and

$$\mathrm{d}\pi_p(\beta'(0)) = (\pi \circ \beta)'(0) = \alpha'(0) = (\pi \circ \gamma)'(0) = \mathrm{d}\pi_p(\gamma'(0))$$

and thus $\beta'(0) = \gamma'(0)$ as $\mathrm{d}\pi_p$ is an isomorphism (as π is a local diffeomorphism). Hence, by uniqueness of geodesics, $\beta = \gamma$ and so β can be defined on all of $\mathbb{R}$. We deduce that $(\widetilde{M}, \tilde{g})$ is complete.

[We see what goes wrong in (b) versus (c): there is no curve lifting the geodesic in $\mathbb{RP}^2$ to $\mathcal{S}^2$ since the map constructed in (b) is not a covering map. In fact, we see that (c) works whenever the surjective local isometry is a covering map.]

2. Let B^n be the unit ball in $\mathbb{R}^n$ and let

$$g = \frac{4 \sum_{i=1}^n \mathrm{d}x_i^2}{(1 - \sum_{i=1}^n x_i^2)^2}.$$

By considering normalized geodesics in (B^n, g) through 0, show that (B^n, g) is complete.

Solution: Let γ be a normalized geodesic in (B^n, g) with $\gamma(0) = 0$. Clearly, if $A \in \mathrm{O}(n)$ then A acts by isometries on (B^n, g), since it preserves B^n, the Euclidean metric and the function $\sum_{i=1}^n x_i^2$ on $\mathbb{R}^n$. Therefore $A \circ \gamma$ is a normalized geodesic also. Hence, we can find $A \in \mathrm{O}(n)$ such that

$$(A \circ \gamma)'(0) = A(\gamma'(0)) = (1, 0, \ldots, 0).$$

Therefore, we now replace γ by $A \circ \gamma$.

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Then, if we choose $B \in \mathrm{O}(n)$ to be

$$B(x_1, \ldots, x_n) = (x_1, -x_2, \ldots, -x_n)$$

we see that $B \circ \gamma(0) = 0$ and $(B \circ \gamma)'(0) = B(\gamma'(0)) = \gamma'(0)$ and so, since $B \circ \gamma$ is also a geodesic, we deduce that $B \circ \gamma = \gamma$. Hence, $\gamma(t) = (x_1(t), 0, \ldots, 0)$.

We see that such a normalized curve must satisfy

$$4(x_1')^2 = (1 - x_1^2)^2$$

Taking square roots, we realize that (up to changing t to $-t$) we require that

$$2x_1' = 1 - x_1^2$$

which we can solve by

$$x_1(t) = \tanh(t/2)$$

since $x_1(0) = 0$ and $x_1'(0) = 1$. This then defines our normalized geodesic and it is clearly defined for all $t \in \mathbb{R}$.

Hence, all normalized geodesics through 0 in (B^n, g) are defined for all $t \in \mathbb{R}$, and thus (B^n, g) is complete by the Hopf–Rinow Theorem.

[The appearance of hyperbolic functions is not surprising given that the metric is of course the hyperbolic metric.]

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Section B

3. Let (N, g) be an oriented $(n+1)$-dimensional Riemannian manifold. Let $f : N \to \mathbb{R}$ be a smooth function and let $h = e^{2f}g$.

   (a) Let ∇^g and ∇^h be the Levi-Civita connections of g and h. Show that

   $$\nabla^h_X Y = \nabla^g_X Y + X(f)Y + Y(f)X - g(X,Y)\nabla^g f$$

   for all vector fields X, Y on N.

   (b) Let M be a connected oriented hypersurface in (N, g) with unit normal vector field ν so that the shape operator satisfies

   $$S_\nu = \lambda\,\mathrm{id}$$

   for a smooth function $\lambda : M \to \mathbb{R}$.

   Show that the shape operator of M in (N, h) satisfies

   $$S_{e^{-f}\nu} = \mu\,\mathrm{id}$$

   for a smooth function $\mu : M \to \mathbb{R}$ which should be identified in terms of λ and f.

   Now let $R > 0$, let

   $$M = \{(x_1, \ldots, x_{n+1}) \in H^{n+1} \ : \ \sum_{i=1}^{n+1} x_i^2 = R^2\}$$

   with its standard orientation and let h be the hyperbolic metric on H^{n+1}.

   (c) Calculate the mean curvature and sectional curvatures of M in (H^{n+1}, h) with its induced metric.

Solution:

(a) We want to show that if we take the formula as a definition of ∇^h then it satisfies the properties of the Levi-Civita connection of h and so has to be ∇^h by uniqueness.

We clearly see that

$$\nabla^h_{aX+bY}Z = a\nabla^h_X Z + b\nabla^h_Y Z$$
$$\nabla^h_X(Y+Z) = \nabla^h_X Y + \nabla^h_X Z$$
$$\nabla^h_X(aY) = a\nabla^h_X Y + X(a)Y$$

since ∇^g has these properties. We then see that

$$\nabla^h_X Y - \nabla^h_Y X = \nabla^g_X Y - \nabla^g_Y X = [X, Y]$$

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since the other terms in ∇^h are symmetric in X, Y.

We then look at

$$\begin{aligned} h(\nabla^h_X Y, Z) + h(Y, \nabla^h_X Z) \\ &= h(\nabla^g_X Y, Z) + h(Y, \nabla^g_X Z) + h(X(f)Y + Y(f)X - g(X,Y)\nabla^g f, Z) \\ &\quad + h(Y, X(f)Z + Z(f)X - g(X,Z)\nabla^g f) \\ &= h(\nabla^g_X Y, Z) + h(Y, \nabla^g_X Z) + X(f)h(Y,Z) + Y(f)h(X,Z) \\ &\quad - g(X,Y)h(\nabla^g f, Z) + X(f)h(Y,Z) + Z(f)h(Y,X) \\ &\quad - g(X,Z)h(Y, \nabla^g f). \end{aligned}$$

We see that

$$Y(f)h(X,Z) - g(X,Z)h(Y,\nabla^g f) = e^{2f}Y(f)g(X,Z) - g(X,Z)e^{2f}g(Y,\nabla^g f) = 0$$

since $g(Y\nabla^g f) = 0$. Similarly,

$$Z(f)h(Y,X) - g(X,Y)h(\nabla^g f, Z) = 0.$$

Hence,

$$\begin{aligned} h(\nabla^h_X Y, Z) + h(Y, \nabla^h_X Z) &= h(\nabla^g_X Y, Z) + h(Y, \nabla^g_X Z) + 2X(f)h(Y,Z) \\ &= e^{2f}\Big(g(\nabla^g_X Y, Z) + g(Y, \nabla^g_X Z)\Big) + 2X(f)e^{2f}g(Y,Z) \\ &= e^{2f}X(g(Y,Z)) + 2X(f)e^{2f}g(Y,Z) \\ &= X(e^{2f}g(Y,Z)) = X(h(Y,Z)). \end{aligned}$$

From this, we deduce that ∇^h is indeed the Levi-Civita connection of h.

[This question shows how the Levi-Civita changes under a *conformal change* of the Riemannian metric. From this, one can deduce how the curvature changes as well, in particular how the scalar curvature changes under a conformal change. This shows how the Yamabe problem of finding a constant scalar curvature metric under a conformal change becomes a PDE on the function f defining the conformal change.]

(b) Clearly

$$h(e^{-f}\nu, e^{-f}\nu) = e^{2f}g(e^{-f}\nu, e^{-f}\nu) = 1$$

and so $e^{-f}\nu$ is a unit normal vector field on M in (N,h).

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We know that, for all tangent vector fields X along M,

$$\begin{aligned}S_{e^{-f}\nu}X &= -(\nabla^h_X(e^{-f}\nu))^{\mathrm{T}}\\ &= -X(e^{-f})\nu^{\mathrm{T}} - e^{-f}(\nabla^h_X\nu)^{\mathrm{T}}\\ &= -e^{-f}(\nabla^g_X\nu + X(f)\nu + \nu(f)X - g(X,\nu)\nabla^g f)^{\mathrm{T}}\\ &= -e^{-f}(\nabla^g_X\nu + \nu(f)X)^{\mathrm{T}},\end{aligned}$$

using the fact that $h(\nu, X) = e^{2f}g(\nu, X) = 0$ and so $\nu^{\mathrm{T}} = 0$. Notice that this means that the tangential parts of vector fields along M are the same in (N, g) and (N, h). Therefore

$$-(\nabla^g_X\nu)^{\mathrm{T}} = S_\nu X = \lambda X$$

by assumption, and so

$$S_{e^{-f}\nu}X = e^{-f}(\lambda - \nu(f))X.$$

Therefore $S_{e^{-f}\nu} = \mu\,\mathrm{id}$ where

$$\mu = e^{-f}(\lambda - \nu(f)).$$

[The hypersurfaces we are considering here are known as *totally umbilic*. This is equivalent to saying that all of the principal curvatures are equal, but they can vary over the hypersurface.]

(c) We want to apply the previous result where $g = g_0$, the Euclidean metric on $N = H^{n+1}$ and $e^{2f} = x_{n+1}^{-2}$ since then $h = e^{2f}g$ is the hyperbolic metric. Therefore,

$$f = -\log x_{n+1}.$$

We see that ν on M in (H^{n+1}, g) is

$$\nu = \frac{1}{R}\sum_{i=1}^{n+1} x_i\partial_i$$

and so

$$\nu(f) = -\frac{1}{R}x_{n+1}\partial_{n+1}\log x_{n+1} = -\frac{1}{R}.$$

We also see that $\nu = \frac{1}{R}\,\mathrm{id}$, identifying tangent vectors on H^{n+1} with vectors in H^{n+1}, and so

$$S_\nu = -\mathrm{d}\nu = -\frac{1}{R}\,\mathrm{id},$$

since we are working with the Euclidean metric. Hence, in the notation of (b),

$$\lambda = -\frac{1}{R} = \nu(f)$$

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and so the shape operator $S_{e^{-f}\nu}$ of M in the hyperbolic space satisfies $S_{e^{-f}\nu} = 0$ by (b) (since $\mu = 0$).

Hence, M is totally geodesic in (H^{n+1}, h) and so has mean curvature $H^M = 0$ and, by the Gauss equation, the sectional curvatures of M are all -1 (the same as the ambient space).

[The hemispheres M in this question are called *horospheres* in the hyperbolic space. In fact, we see that the functions λ and μ are both constant which is not a coincidence, since they have to be if the ambient manifold N has a Riemannian metric with constant sectional curvature.]

4. (a) Let $(M.g)$ be a complete Riemannian manifold with non-positive sectional curvature, let p, q be points in M and let α be a curve in M from p to q.

   Show that there is a unique geodesic γ in (M, g) from p to q which is homotopic to α.

   (b) Let (M, g) be an oriented even-dimensional manifold with positive sectional curvature and let $\gamma : \mathcal{S}^1 \to (M, g)$ be a closed geodesic.

   Show that there is a closed curve $\alpha : \mathcal{S}^1 \to (M, g)$ homotopic to γ such that $L(\alpha) < L(\gamma)$.

Solution:

(a) The universal cover $(\widetilde{M}, \tilde{g})$ of (M, g) with the covering metric is complete with non-positive sectional curvature.

Let $\pi : \widetilde{M} \to M$ be the covering map and let $x \in \pi^{-1}(p)$. By the curve-lifting property there is a curve $\tilde{\alpha}$ in $\widetilde{M}$ with $\tilde{\alpha}(0) = x$ and $\pi \circ \tilde{\alpha} = \alpha$. Let $y \in \pi^{-1}(q)$ be the other endpoint of $\tilde{\alpha}$.

By the Cartan–Hadamard Theorem, since $\widetilde{M}$ is simply connected, the exponential map $\exp_x : T_x\widetilde{M} \to \widetilde{M}$ is a diffeomorphism. Hence, there exists a unique $X \in T_x\widetilde{M}$ so that $\exp_x(X) = y$, and thus there is a unique geodesic $\tilde{\gamma}$ from x to y given by $\tilde{\gamma}(t) = \exp_x(tX)$. Since $\widetilde{M}$ is simply connected $\tilde{\gamma}$ is homotopic to $\tilde{\alpha}$ with the fixed endpoints.

Let $\gamma = \pi \circ \tilde{\gamma}$. Then γ is a geodesic from p to q in (M, g) since π is a local isometry, and γ is homotopic to α since the lift $\tilde{\gamma}$ has the same endpoints as the lift $\tilde{\alpha}$ of α.

Suppose that β is another geodesic from p to q homotopic to α. Then by the curve-lifting property there is a curve $\tilde{\beta}$ in $\widetilde{M}$ with $\tilde{\beta}(0) = x$ and $\pi \circ \tilde{\beta} = \beta$. Since π is a local isometry, $\tilde{\beta}$ is a geodesic and, since β is homotopic to α, the other endpoint of $\tilde{\beta}$ is the same as that of $\tilde{\alpha}$, i.e. y. Thus $\tilde{\gamma} = \tilde{\beta}$ by uniqueness of the geodesic $\tilde{\gamma}$, and hence $\beta = \gamma$, and so γ is unique.

[In particular, this result shows that there is a unique geodesic representing each element of the fundamental group of M, by taking $p = q$.]

(b) Write $\gamma(t)$ for $e^{it} \in \mathcal{S}^1$.

Let $p = \gamma(0) \in \gamma(\mathcal{S}^1) \subseteq M$ and suppose M has dimension $2n$. Consider the parallel transport $\tau_\gamma : T_pM \to T_pM$ around γ. Then τ_γ is an orientation-preserving isometry and so defines an element in $\mathrm{SO}(2n)$.

We know that γ' is a parallel vector field so $X = \gamma'(0)$ is fixed by τ_γ. Hence, $\tau_\gamma : \mathrm{Span}\{X\}^\perp \to \mathrm{Span}\{X\}^\perp$ defines an element in $\mathrm{SO}(2n-1)$, and thus has must have 1 as an eigenvalue (since complex eigenvalues occur in complex conjugate pairs and their product is 1, and the product of all eigenvalues is 1, so there is an odd number of real eigenvalues whose product is 1, all of modulus 1, and so at least one must be 1).

Therefore, there exists a unit eigenvector Y of τ_γ of eigenvalue 1 so that Y is orthogonal to X. Let $Y(t)$ be the parallel vector field along γ with $Y(0) = Y$. Notice that $Y(2\pi) = Y = Y(0)$ since Y is fixed by τ_γ. Consider the variation f of γ given by

$$f(s,t) = \exp_{\gamma(t)}(sY(t))$$

which gives a well-defined map $f : (-\epsilon, \epsilon) \times \mathcal{S}^1 \to M$ (since $Y(2\pi) = Y(0) = Y$).

Note that

$$g(\gamma', Y)' = g(\gamma'', Y) + g(\gamma', Y') = 0$$

so $g(\gamma', Y) = 0$ along γ (since this holds at $t = 0$). We see that

$$V_f(t) = \frac{\partial f}{\partial s}|_{s=0} = \mathrm{d}(\exp_{\gamma(t)})_0(Y(t)) = Y(t)$$

and so $V_f' = Y' = 0$ as Y is parallel along γ.

By the First Variation formula

$$\frac{1}{2}E_f'(0) = -\int_0^{2\pi} g(V_f, \nabla_{\gamma'}\gamma')\mathrm{d}t - g(V_f(0), \gamma'(0)) + g(V_f(2\pi), \gamma'(L)) = 0.$$

By the Second Variation formula we have that:

$$\begin{aligned}\frac{1}{2}E_f''(0) &= \int_0^{2\pi} g(V_f', V_f') - R(V_f, \gamma', \gamma', V_f)\mathrm{d}t \\ &\quad - g(\nabla_{\frac{\partial f}{\partial s}}\frac{\partial f}{\partial s}(0,0), \gamma'(0)) + g(\nabla_{\frac{\partial f}{\partial s}}\frac{\partial f}{\partial s}(0,2\pi), \gamma'(2\pi)) \\ &= \int_0^{2\pi} -R(V_f, \gamma', \gamma', V_f)\mathrm{d}t\end{aligned}$$

since the values of $g(\nabla_{\frac{\partial f}{\partial s}}\frac{\partial f}{\partial s}(0,t),\gamma'(t))$ for $t=0$ and $t=2\pi$ are equal and $V_f' = Y' = 0$. Therefore,

$$\frac{1}{2}E_f''(0) = -\int_0^{2\pi} K(\gamma',Y)|Y|^2|\gamma'|^2\mathrm{d}t = -\int_0^{2\pi} K(\gamma',Y)|\gamma'|^2\mathrm{d}t < 0$$

as the sectional curvature $K>0$ on (M,g).

Hence, 0 is a strict local maximum of E_f, hence for s sufficiently small the curve $\alpha = f(s,.) : \mathcal{S}^1 \to (M,g)$ has $L(\alpha) < L(\gamma)$ and α is homotopic to γ.

[The obvious example here is the 2-sphere: start with the equator (a closed geodesic), then you can push it up or down so that it becomes another line of latitude and clearly the length goes down.

The number of directions in which you can decrease the length of the closed geodesic is called the (Morse) index, and is related to *Morse theory*.

The index measures "how unstable" the geodesic is as a critical point for the length functional. This is clearly related to the Second Variation formula.

A result of Cartan states that on a compact Riemannian manifold every element of the fundamental group can be represented by a closed geodesic which minimizes length in its homotopy class. What we have shown implies that if we have a compact oriented even-dimensional manifold (M,g) with positive sectional curvature, any closed geodesic can be made shorter in its homotopy class. Putting this together with Cartan's result, we deduce that (M,g) is simply connected which is, of course, part (a) of Synge's Theorem.

Notice that the assumption that (M,g) is compact is important, since there are obvious counterexamples – take the paraboloid in $\mathbb{R}^3$, which is even, oriented and has positive sectional curvature, and remove the origin, so that it now has fundamental group $\mathbb{Z}$ but there is definitely no closed geodesic in the paraboloid minus the origin minimizing length in any non-trivial homotopy class.]

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5. (a) Let $n, m \in \mathbb{N}$. Show that $\mathcal{S}^n \times \mathcal{S}^m$ admits a Riemannian metric of positive Ricci curvature if and only if $n \geq 2$ and $m \geq 2$.

(b) Let G be a connected Lie group with identity e which admits a bi-invariant Riemannian metric. Suppose that the centre of the Lie algebra $\mathfrak{g} = T_eG$ is trivial.

Show that G and its universal cover are compact, and hence that $\mathrm{SL}(n, \mathbb{R})$ does not admit a bi-invariant metric for $n \geq 2$.

[*You may assume that the results of Problem sheet 3 Question 4 extend to any Lie group with a bi-invariant Riemannian metric.*]

(c) Show that $\mathbb{RP}^2 \times \mathbb{RP}^2$ does not admit a Riemannian metric of positive sectional curvature.

[*Hint: You may want to think about the orientable double cover.*]

Solution:

(a) Any Riemannian metric of positive Ricci curvature on $\mathcal{S}^n \times \mathcal{S}^m$ would have to be complete since $\mathcal{S}^n \times \mathcal{S}^m$ is compact (by the Hopf–Rinow Theorem), and there is definite positive lower bound on the Ricci curvature since $\mathcal{S}^n \times \mathcal{S}^m$ is compact. Therefore, we may apply the Bonnet–Myers Theorem and deduce that the fundamental group of $\mathcal{S}^n \times \mathcal{S}^m$ must be finite, which then forces both $n \geq 2$ and $m \geq 2$ (since otherwise there is at least a $\mathbb{Z}$ factor in the fundamental group from an $\mathcal{S}^1$ factor).

We see from Problem Sheet 3 that if $n \geq 2$ and $m \geq 2$, if we take the round metric on $\mathcal{S}^n$ and $\mathcal{S}^m$ then the Ricci curvature of the product metric on $\mathcal{S}^n \times \mathcal{S}^m$ satisfies

$$\mathrm{Ric}((X_1, X_2), (Y_1, Y_2)) = (n-1)g_{\mathcal{S}^n}(X_1, Y_1) + (m-1)g_{\mathcal{S}^m}(X_2, Y_2).$$

Therefore,

$$\mathrm{Ric}((X_1, X_2), (X_1, X_2)) = (n-1)|X_1|^2 + (m-1)|X_2|^2 > 0$$

whenever $(X_1, X_2) \neq (0, 0)$. Thus the product metric has positive Ricci curvature.

(b) Let g be a bi-invariant Riemannian metric on G. By Problem Sheet 3, we know that the sectional curvature K at e satisfies

$$K(X, Y) = \frac{1}{4}|[X, Y]|^2 \geq 0$$

and equals zero if and only if X, Y commute in $\mathfrak{g} = T_eG$. Let $\{E_1, \ldots, E_n\}$ be an orthonormal basis for $\mathfrak{g}$. Then

$$\mathrm{Ric}(E_i, E_i) = \sum_{k=1}^{n} R(E_i, E_k, E_k, E_i) = \sum_{k=1}^{n} K(E_i, E_k) \geq 0$$

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with equality if and only if $K(E_i, E_k) = 0$ for all k.

However, if $K(E_i, E_k) = 0$ for all k, we must have that $[E_i, E_k] = 0$ for all k, which means that E_i lies in the centre of $\mathfrak{g}$, which is a contradiction to the assumption that the centre is trivial. Hence, $\mathrm{Ric}(E_i, E_i) > 0$ for all i and thus $\mathrm{Ric}(X, X) \geq \delta > 0$ for all $X \in \mathfrak{g}$.

Moreover, since Ric is defined by its values at the identity as g is bi-invariant, we deduce that $\mathrm{Ric} \geq \delta > 0$ on (G, g). Hence, we may apply Bonnet–Myers and deduce that G is compact with finite fundamental group, so that its universal cover is also compact, if we can show that (G, g) is complete.

From Problem Sheet 3, we know that for any left-invariant vector field X on G we have that

$$\nabla_X X = \frac{1}{2}[X, X] = 0.$$

Therefore, integral curves of X are geodesics in (G, g). An integral curve α of X with $\alpha(0) = e$ is defined by

$$\alpha'(t) = X(\alpha(t)).$$

Since X and g are left-invariant, $|X(\alpha(t))| = |X(\alpha(0))| = |X(e)|$ is independent of t, and so the differential equation has a long-time solution, i.e. $\alpha(t)$ is defined for all t (when G is a matrix Lie group as in Problem Sheet 3, $\alpha(t) = \exp(tX(e)) = e^{tX(e)}$). Since $X(e)$ is arbitrary for left-invariant vector fields, we deduce that all geodesics starting at e are integral curves of left-invariant vector fields (by uniqueness of geodesics), and defined for all time, and so (G, g) is complete by the Hopf–Rinow Theorem.

Hence G and its universal cover are indeed compact by Bonnet–Myers.

Since $\mathrm{SL}(n, \mathbb{R})$ is non-compact (it contains all diagonal matrices with entries e^t, e^{-t} and 1 otherwise for all $t > 0$) and has Lie algebra

$$\mathfrak{sl}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) \,:\, \operatorname{tr} X = 0\}$$

which has trivial centre, we deduce that $\mathrm{SL}(n, \mathbb{R})$ cannot admit a bi-invariant metric.

[This result is key to classifying the Lie groups which admit bi-invariant Riemannian metrics, since any compact Lie group admits a bi-invariant Riemannian metric.]

(c) Suppose that g is a Riemannian metric of positive curvature on $M = \mathbb{RP}^2 \times \mathbb{RP}^2$. Then g is complete as M is compact (by the Hopf–Rinow Theorem) and there is $\delta > 0$ so that $K \geq \delta$ by compactness of M as well.

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Let $\overline{M}$ be the oriented double cover of M with the covering metric $\overline{g}$. Since (M, g) is complete and $\pi : (\overline{M}, \overline{g}) \to (M, g)$ is a surjective local isometry which is a covering map, we deduce that $(\overline{M}, \overline{g})$ is complete (we only need that $\overline{M}$ is compact to deduce this in fact) and has $\overline{K} \geq \delta > 0$.

Since $(\overline{M}, \overline{g})$ is compact, orientable and even-dimensional with $\overline{K} > 0$, we deduce from Synge's Theorem that $\overline{M}$ is simply connected. Since $\pi : \overline{M} \to M$ is a double cover, we conclude that the fundamental group of M must be contained in $\mathbb{Z}_2$. This is a contradiction since the fundamental group of $M = \mathbb{RP}^2 \times \mathbb{RP}^2$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

[This result is perhaps surprising given that the Hopf conjecture that $\mathcal{S}^2 \times \mathcal{S}^2$ does not admit a Riemannian metric with positive sectional curvature remains open. In general, the question of which manifolds admit Riemannian metrics with positive sectional curvature is definitely amongst the most difficult in geometry and topology.]

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Section C

6. Determine whether each of the following statements is true or false, and give a proof or counterexample as appropriate.

   (a) The unitary group $\mathrm{U}(m)$ admits a Riemannian metric with strictly positive Ricci curvature for some $m > 1$.

   (b) The manifold $\mathcal{S}^n \times \mathcal{S}^m$ admits a Riemannian metric with non-positive sectional curvature if and only if $n = m = 1$.

   (c) Euclidean space $\mathbb{R}^n$ admits a constant curvature 1 Riemannian metric for any $n > 1$.

   (d) If K is the Klein bottle then $K \times \mathcal{S}^n$ admits a Riemannian metric with positive sectional curvature for any $n > 1$.

   (e) Complex projective space $\mathbb{CP}^n$ admits a constant curvature 1 Riemannian metric if and only if $n = 1$.

   [*Hint: You may assume that* $\pi_1(\mathbb{CP}^n) = \{1\}$ *and* $H^2(\mathbb{CP}^n) \neq 0$ *for all* n.]

Solution:

(a) This is false.

The unitary group $\mathrm{U}(m)$ has infinite fundamental group for any m since it is a semi-direct product of $\mathrm{U}(1) \cong \mathcal{S}^1$ and $\mathrm{SU}(m)$, and so its fundamental group contains a copy of $\mathbb{Z}$.

Since any Riemannian metric on $\mathrm{U}(m)$ is complete, because $\mathrm{U}(m)$ is compact, we deduce that $\mathrm{U}(m)$ cannot admit a Riemannian metric with positive Ricci curvature by the Bonnet–Myers Theorem for any $m \geq 1$.

(b) This is true.

If $M = \mathcal{S}^n \times \mathcal{S}^m$ admits a Riemannian metric with non-positive sectional curvature it must be complete (as M is compact) and so by the Cartan–Hadamard Theorem we deduce that the universal cover of M must be $\mathbb{R}^{n+m}$.

However, if both of n, m are greater than 1, then M is simply connected, so it is its own universal cover, and M is not diffeomorphic to $\mathbb{R}^{n+m}$ because it is compact.

If $n = 1$ and $m > 1$, say, then the universal cover of M is $\mathbb{R} \times \mathcal{S}^m$ which is still not diffeomorphic to $\mathbb{R}^{m+1}$ because $H^m(M) \neq 0$.

If $n = m = 1$ then $M = \mathcal{S}^1 \times \mathcal{S}^1$, which we know admits a flat metric.

(c) This is true.

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Given any $n > 1$ can embed $\mathbb{R}^n$ in $(\mathcal{S}^n, g)$ with the round metric as $\mathcal{S}^n \setminus \{N\}$, where N is the North pole. Then the induced Riemannian metric on $\mathbb{R}^n$ will have constant curvature 1 (since $n > 1$).

[Of course, we cannot have a *complete* Riemannian metric with constant curvature 1 on $\mathbb{R}^n$ by the classification of space forms.]

(d) This is false.

We see that if we take $n = 3$ then $K \times \mathcal{S}^3$ is compact, odd-dimensional and not orientable. Therefore, by Synge's Theorem, it cannot admit a Riemannian metric with positive sectional curvature.

[You may want to think about what you can say when n is even in this part.]

(e) This is true.

For all n, we have that $\mathbb{CP}^n$ is simply connected and $H^2(\mathbb{CP}^n) \neq 0$ by the hint. Hence $\mathbb{CP}^n$ is not diffeomorphic to $\mathcal{S}^{2n}$ for $n > 1$. Therefore, by the classification of space forms (since $\mathbb{CP}^n$ is compact and so any Riemannian metric on it would be complete) we deduce that $\mathbb{CP}^n$ cannot admit a constant curvature 1 metric for $n > 1$.

For $n = 1$, we know that $\mathbb{CP}^1$ is diffeomorphic to $\mathcal{S}^2$, and so does indeed admit a constant curvature 1 metric.

[In case of interest, you may want to try to prove the topological claims about $\mathbb{CP}^n$ in the hint, though that is not part of the Riemannian Geometry course.]

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