B8.3: Mathematical Modelling of Financial Derivatives —Exercises—

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Exercise Sheet 4

Part A

1. The instalment option has the same payoff as that of a vanilla call or a put option; it may be European or American. Its unusual feature is that, as well as paying the initial premium, the holder must pay 'instalments' during the life of the option. The instalments may be paid either continuously or discretely. The holder can choose at any time to stop paying the instalments, at which point the contract is cancelled and the option ceases to exist.

   Assume instalments are paid continuously at a rate $L(t)$ per unit time. Derive the differential equation satisfied by the option price. What new constraint must it satisfy?

Part B

1. Assume that the USD/GBP exchange rate, X_t, evolves according to the SDE

$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t.$$

   (a) Today's exchange rate is X_0, find the expected USD/GBP exchange rate $\mathbb{E}[X_T]$ at time $T > 0$.

   (b) Find the SDE followed by the GBP/USD exchange rate, i.e., find the dynamics for $Y_t = 1/X_t$.

   (c) Given that $Y_0 = 1/X_0$ today, find the expected GBP/USD exchange rate $\mathbb{E}[Y_T]$ at time $T > 0$.

   (d) Show that, although $X_T\, Y_T = 1$ for any $T > 0$,

$$\mathbb{E}[X_T]\,\mathbb{E}[Y_T] = e^{\sigma^2 T}.$$

2. A European log-put option has the payoff

$$V_T = \big(-\log(S_T/K)\big)^+$$

   (a) Show that if S_u evolves as

$$\frac{dS_u}{S_u} = r\,du + \sigma\,dW_u,\ t < u \le T, \quad S_t = S,$$

   then

$$\text{prob}(S_T < K) = \mathrm{N}(-d_-), \quad d_- = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2\,(T-t)}}.$$

   (b) Assuming the underlying share pays no dividends, show that the Black–Scholes formula for the log-put is

$$V(S,t) = e^{-r(T-t)}\sqrt{\sigma^2\,(T-t)}\,\Big(d_-\,\mathrm{N}(-d_-) - e^{-\frac{1}{2}d_-^2}/\sqrt{2\,\pi}\Big).$$

3. An investor has the choice of investing their wealth of 1 unit of currency in either a risky asset whose price evolves as

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t,\ t > 0, \quad S_0 = 1,$$

where $\sigma > 0$, or in a risk-free bond whose price evolves as

$$\frac{dB_t}{B_t} = r\,dt,\ t > 0, \quad B_0 = 1,$$

where $0 < r < \mu - \frac{1}{2}\sigma^2$. The investment horizon is $[0, T]$. The investor decides to invest their funds in the risky asset, but is worried that when they withdraw the funds, at time T, the risk-free bonds may have outperformed the risky assets. So they consider the possibility of purchasing a put option with maturity T to protect themselves against this possibility. (They borrow money to buy the put.)

   (a) What is the probability of the risky asset underperforming the risk-free one, i.e., what is the probability that $S_T < e^{rT}$?

   (b) What happens to this probability as $T \to \infty$?

   (c) What should the strike of the put be so that the investor is fully insured against the possibility of underperformance?

   (d) What happens to the price of the insurance as $T \to \infty$?

4. Let T_1 and T_2 be given times with $0 < T_1 < T_2$ and let $\alpha > 0$ be a given constant. A forward-start put is a European put option written on an asset whose price is S_t, but where the strike is not given at time zero, rather it is set equal to $\alpha\,S_{T_1}$, where S_{T_1} is the share price at time T_1. Find the option price and Δ for $T_1 < t < T_2$ and then for $0 \le t \le T_1$.

Part C

1. Assume the stock price S follows the usual Black–Scholes dynamics and that there are no dividend payments. Let $0 < T_1 < T_2$ and $K > 0$. A derivative security with the following properties is written on a share (which does not pay any dividends between time $t = 0$ and $t = T_2$). If at time T_1 the share price is greater than or equal to K, $S_{T_1} \geq K$, then the derivative security becomes a European call option with strike S_{T_1} and expiry date T_2. If $S_{T_1} < K$, it becomes a European put option with strike S_{T_1} and expiry date T_2. What is the price of this derivative for $T_1 < t < T_2$ and for $0 \leq t \leq T_1$.