Geometric Group Theory

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An inspirational quotation

Henri Poincaré argued that the understanding of a structure means the understanding of the group of transformations preserving it, and that the concept of group is innate, and key to reasoning itself.

Henri Poincaré: "The object of geometry is the study of a particular 'group'; but the general concept of group pre-exists in our minds, at least potentially. It is imposed on us not as a form of our senses, but as a form of our understanding.

Only, from among all the possible groups, that must be chosen which will be, so to speak, the standard to which we shall refer natural phenomena."

Graphs of groups and actions on trees

In the last lecture we proved the following result.

Theorem

$H = \pi_1(G, Y, a_0)$ acts on a tree T without inversions, such that

1. *The quotient graph $H \backslash T$ can be identified with Y;*
2. *Let $q : T \to Y$ be the quotient map:*
    - a. *For all $v \in V(T)$, $\mathrm{Stab}_H(v)$ is a conjugate in H of $G_{q(v)}$;*
    - b. *For all $e \in E(T)$, $\mathrm{Stab}_H(e)$ is a conjugate in H of $G_{q(e)}$.*

We denote the tree thus obtained $\mathcal{T}(G, Y, a_0)$ and we call it the universal covering tree or the Bass–Serre tree of the graph of groups (G, Y).

Graphs of groups and actions on trees

Conversely, if a group Γ acts on a tree T with quotient Y then there exists a graph of groups (G, Y) such that $\Gamma \simeq \pi_1(G, Y, a_0)$.

Indeed, suppose $\Gamma \curvearrowright T$, $Y = T/\Gamma$ and $p : T \to Y$.

Let $X \subset S \subset T$ be such that $p(X)$ is a maximal tree of Y, $p(S) = Y$ and $p\big|_{\text{edges of } S}$ is 1-to-1.

Notation: If v is a vertex of Y and e is an edge of Y, let

- v^X be the vertex of X such that $p(v^X) = v$;
- e^S be the edge of S such that $p(e^S) = e$.

A graph of groups with graph Y:

1. The map G:
    - Let $G_v = \mathrm{Stab}_\Gamma(v^X)$;
    - Let $G_e = \mathrm{Stab}_\Gamma(e^S)$.

Graphs of groups and actions on trees

1. The map G:
   - Let $G_v = \mathrm{Stab}_\Gamma(v^X)$;
   - Let $G_e = \mathrm{Stab}_\Gamma(e^S)$.
2. For each edge e, we define $\alpha_e : G_e \to G_{t(e)}$: For all $x \in V(S)$, define

$$g_x = \begin{cases} 1 & \text{if } x \in V(X) \\ \text{some } g_x \text{ such that } g_x x \in V(X) & \textit{otherwise}. \end{cases}$$

   Define $\alpha_e : G_e \to G_{t(e)}, \alpha_e(g) = g_{t(e)} g g_{t(e)}^{-1}$.

We can define a homomorphism $\varphi : F(G, Y) \to \Gamma$ by:

- $\forall a \in V(Y)$, $\varphi\big|_{G_a} = \mathrm{incl}_{G_a}$;
- $\forall e \in E(Y)$, $e = [y, x]$, $\varphi(e) = g_y g_x^{-1}$.

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It satisfies the relations:

$$\varphi(\bar{e}) = g_x g_y^{-1} = (g_y g_x^{-1})^{-1} = \varphi(e)^{-1}$$

$$\varphi(e\alpha_e(g)e^{-1}) = (g_y g_x^{-1})(g_x g g_x^{-1})(g_x g_y^{-1}) = g_y g g_y^{-1} = \varphi(\alpha_{\bar{e}}(g))$$

Also, $\forall e \in p(X)$, $\varphi(e) = 1$. Hence, φ defines a homomorphism

$$\bar{\varphi} : \pi_1(G, Y, p(X)) \simeq \pi_1(G, Y, a_0) \to \Gamma$$

Graphs of groups and actions on trees

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Theorem

The homomorphism $\bar{\varphi}$ is an isomorphism. If $\widetilde{\mathcal{T}} = \mathcal{T}(G, Y, a_0)$ is the universal covering tree of (G, Y) then there exists a graph isomorphism $f : \widetilde{\mathcal{T}} \to T$ such that $\forall g \in \pi_1(G, Y, a_0)$, $\forall v \in V(\widetilde{\mathcal{T}})$,

$$f(g \cdot v) = \bar{\varphi}(g) \cdot f(v).$$

Proof: Not provided and non-examinable.

Theorem

Let $\Gamma = \pi_1(G, Y, a_0)$. If $B \leq \Gamma$ then there exists (H, Z) a graph of groups such that $B = \pi_1(H, Z, b_0)$ and

- *for all $v \in V(Z)$, $H_v \leq gG_a g^{-1}$ for some $a \in V(Y)$, $g \in \Gamma$;*
- *for all $e \in E(Z)$, $H_e \leq \gamma G_y \gamma^{-1}$, for some $y \in E(Y)$, $\gamma \in \Gamma$.*

Proof.

Γ acts on a tree T with quotient a graph of groups (G, Y). The subgroup B acts on T, $\mathrm{Stab}_B(v) \leq \mathrm{Stab}_\Gamma(v)$ for all $v \in V(T)$ and $\mathrm{Stab}_B(e) \leq \mathrm{Stab}_\Gamma(e)$ for all $e \in E(T)$. □

NB It may be that, while Y is finite, Z is infinite.

Subgroups

Theorem (Kurosh)

Suppose $G = G_1 * ... * G_n$. If $H \leq G$ then

$$H = (*_{i \in I} H_i) * F$$

where I is finite or countable, F is a free group and the H_i are subgroups of conjugates of G_j.

Unique decomposition I

We say that G is indecomposable if $G \neq A * B$.

Theorem (Grushko)

Suppose G is finitely generated. There exists indecomposable $G_1, \dots, G_k$ such that

$$G = G_1 * \dots * G_k * F_n$$

Moreover, if there exist other indecomposable $H_1, \dots, H_m$ such that

$$G = H_1 * \dots * H_m * F_r$$

then $m = k$, $r = n$ and, after reordering, H_i is conjugate to G_i for all i.

Unique decomposition II

Theorem (Dunwoody)

Suppose Γ is finitely presented. Then Γ can be written as $\pi_1(G, Y, a_0)$ where (G, Y) is a finite graph of groups such that all edge groups are finite and all the G_v do not split over finite groups.

Theorem (Stallings)

A group Γ does not split over finite groups if and only if it is one-ended.

A group Γ is one-ended if any (every) Cayley graph cannot be disconnected by removing a compact subset.

Quasi-isometry

Definition

Let $f : X \to Y$ be a map between metric spaces.

1. We say that f is an (L, A)-quasi-isometric embedding if for some constants $L \geq 1$, $A \geq 0$ and for all $x_1, x_2 \in X$ we have

$$\frac{1}{L} d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + A$$

   It is called a quasi-isometry if moreover we have that for all $y \in Y$, there exists some $x \in X$ such that $d(y, f(x)) \leq A$.

2. If $I \subseteq \mathbb{R}$ is an interval, then an (L, A)-quasi-isometric embedding $\gamma : I \to X$ is called an (L, A)-quasi-geodesic.

3. If there exists a quasi-isometry $f : X \to Y$ between two metric spaces then we say that X and Y are quasi-isometric.

Quasi-isometry

Examples

1. $\mathbb{Z}^2$ *and* $\mathbb{R}^2$ *are quasi-isometric.*
2. *If* G *is a finitely generated group with finite generating sets* S, S' *then the Cayley graphs* $\Gamma(S, G)$, $\Gamma(S', G)$ *are quasi-isometric.*
3. *If* T_n *is the n-valent tree, then* $T_n \sim T_3$ *for all* $n \in \mathbb{N}$.