Geometric Group Theory

Cornelia Druțu

University of Oxford

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Quotations

Ralph Waldo Emerson: “Life is a journey, not a destination.”

Donald Knuth: “It would be nice if we could design a virtual reality in Hyperbolic space, and meet each other there.”

Quasi-isometry

Definition

Let $f : X \to Y$ be a map between metric spaces.

1. We say that f is an (L, A)-quasi-isometric embedding if for some constants $L \geq 1$, $A \geq 0$ and for all $x_1, x_2 \in X$ we have

$$\frac{1}{L} d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + A$$

   It is called a quasi-isometry if moreover we have that for all $y \in Y$, there exists some $x \in X$ such that $d(y, f(x)) \leq A$.

2. If $I \subseteq \mathbb{R}$ is an interval, then an (L, A)-quasi-isometric embedding $\gamma : I \to X$ is called an (L, A)-quasi-geodesic.

3. If there exists a quasi-isometry $f : X \to Y$ between two metric spaces then we say that X and Y are quasi-isometric.

Quasi-isometry

Examples

1. $\mathbb{Z}^2$ *and* $\mathbb{R}^2$ *are quasi-isometric.*
2. *If* G *is a finitely generated group with finite generating sets* S, S' *then the Cayley graphs* $\Gamma(S, G)$, $\Gamma(S', G)$ *are quasi-isometric.*
3. *If* T_n *is the n-valent tree, then* $T_n \sim T_3$ *for all* $n \in \mathbb{N}$.

The following theorem implies the first example above and is our main source of quasi-isometries.

Quasi-isometry

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Theorem (Milnor–Švarc)

Suppose G acts by isometries on a metric space X such that

1. a. *X is geodesic;*
   b. *X is proper (closed balls are compact);*
2. *the action is*
   a. *properly discontinuous: i.e. given a compact $K \subseteq X$, the set $\{g \in G : g(K) \cap K \neq \emptyset\}$ is finite;*
   b. *cocompact: i.e. there exists a compact $K' \subseteq X$ such that $GK' = X$;*

then G is finitely generated and every orbit map $G \to X$, $g \mapsto g \cdot x_0$ is a quasi-isometry when G is endowed with a word metric.

Proof is non-examinable.

Quasi-isometry

Corollary

Suppose G is a finitely generated group with some word metric.

1. *If $H \leq G$ is a finite index subgroup then H is quasi-isometric to G.*
2. *If $N \lhd G$ is a finite normal subgroup then G is quasi-isometric to G/N.*
3. *Suppose M is a compact Riemannian manifold. Then $\pi_1(M)$ is quasi-isometric to the universal cover $\tilde{M}$.*

Exercise: Suppose a group G is quasi-isometric to a finitely presented group H. Then G is finitely presented.

Hyperbolic space

Definition

Let X be a geodesic metric space. Given $A \subseteq X$ and $r > 0$, the *r-neighbourhood of A in X* is the subset

$$\mathcal{N}_r(A) = \{x \in X : d(x, A) < r\} \subseteq X.$$

Let $x, y, z \in X$. A *geodesic triangle* $[x, y, z]$ in X is the union of three geodesic paths $[x, y], [y, z], [z, x]$:

$$[x, y, z] = [x, y] \cup [y, z] \cup [z, x]$$

We say that a geodesic triangle $[x, y, z]$ is δ-*slim* for some $\delta \geq 0$ if each side is within a δ-neighbourhood of the other two sides: for example $[x, y] \subseteq \mathcal{N}_\delta([y, z] \cup [z, x])$.

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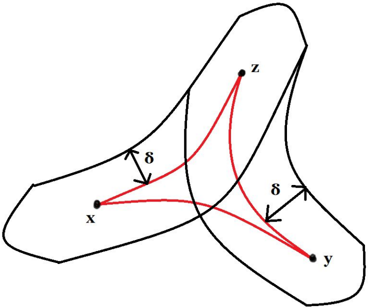
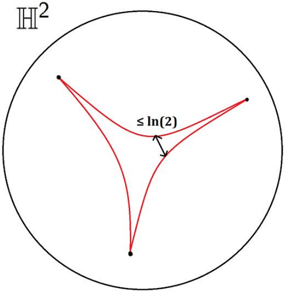


We say that X is δ-hyperbolic if every geodesic triangle is δ-slim.

Examples of δ-hyperbolic spaces

Examples

1. *Any tree is* 0*-hyperbolic.*
2. *Any metric space X with finite diameter is δ-hyperbolic (for example take δ to be the diameter of X).*
3. $\mathbb{R}^2$ *is not hyperbolic.*
4. $\mathbb{H}^2$ *is* $\ln(2)$*-hyperbolic:*



δ-thin geodesic triangles

Let $\Delta = [x, y, z]$ be a geodesic triangle in X. There is a unique metric tree T_Δ (a 'tripod' if x, y, z pairwise distinct) with endpoints x', y', z' (corresponding to x, y, z respectively) s. t. there exists an onto map $f_\Delta : \Delta \to T_\Delta$ which restricts to an isometry from each side $[x, y], [y, z], [z, x]$ to the corresponding side $[x', y'], [y', z'], [z', x']$.

Definition

A geodesic triangle $\Delta = [x, y, z]$ in X is δ-thin if for every $t \in T_\Delta$, $diam(f_\Delta^{-1}(t)) \leq \delta$.

Theorem

Let X be a geodesic metric space. X is δ-hyperbolic if and only if there exists some $\delta' \geq 0$ such that every geodesic triangle in X is δ'-thin.

Proof: Exercise 5, Ex Sheet 4.

δ-hyperbolic spaces

Lemma

Suppose X is a geodesic δ-hyperbolic space (i.e. all geodesic triangles are δ-slim) and let $x_0, ..., x_n \in X$. Then

$$[x_0, x_n] \subseteq \mathcal{N}_{(\log_2(n)+1)\delta}([x_0, x_1] \cup ... \cup [x_{n-1}, x_n])$$

Proof: Choose k such that $2^{k-1} < n \leq 2^k$. We will prove by induction on k that

$$[x_0, x_n] \subseteq \mathcal{N}_{k\delta}([x_0, x_1] \cup ... \cup [x_{n-1}, x_n])$$

For $k = 1$ this is obvious. Assume true for $k - 1$ and let $p \in [x_0, x_n]$. Let $m = 2^{k-1}$. By hyperbolicity, there exists $p_1 \in [x_0, x_m] \cup [x_m, x_n]$ such that $d(p_1, p) \leq \delta$. By the inductive hypothesis

$$d(p_1, [x_0, x_1] \cup ... \cup [x_{n-1}, x_n]) \leq (k - 1)\delta$$

and the result follows. □

Proposition (Morse lemma)

Let X be a δ-hyperbolic metric space. For any $\lambda \geq 1$ and $\mu \geq 0$, there exists some $M = M(\lambda, \mu)$ such that if

- *$\alpha : [u, v] \to X$ is a (λ, μ)-quasi-geodesic with endpoints $x = \alpha(u)$, $y = \alpha(v)$;*
- *$\gamma = [x, y]$ is a geodesic with the same endpoints as α;*

then $\alpha \subseteq \mathcal{N}_M(\gamma)$ and $\gamma \subseteq \mathcal{N}_M(\alpha)$.

Proof

Without loss of generality we can assume α is continuous and such that

$$\mathrm{length}(\alpha([t, s])) \leq \lambda d(\alpha(t), \alpha(s)) + \mu$$

for every $t, s \in [u, v]$ (see Exercise 3 on Ex. Sheet 4).