Axiomatic Set Theory: Problem sheet 3

A.

1. Assuming (as was shown in the lectures), that $a \in L \to \bigcup a \in L$ and $a \in L \to \wp a \cap L \in L$, verify carefully that $\langle L, \in\rangle \models$ Union, Powerset.

Let a be an element of L and let $b = \bigcup a$.

Then $\langle V, in\rangle \models (\forall c \in b\, \exists d \in a\, c \in d) \wedge (\forall c \in a\, \forall d \in c\, d \in b)$.

Now this statement is Σ_0, so absolute between transitive classes, so $\langle L, in\rangle \models (\forall c \in b\, \exists d \in a\, c \in d) \wedge (\forall c \in a\, \forall d \in c\, d \in b)$.

So the Axiom of Unions is true in L.

Suppose that $a \in L$ and $b = L \cap \wp a$.

Then using the fact that L is transitive, $\langle V, \in\rangle \models (\forall c \in L\, (c \in b \to \forall d \in L\, (d \in c \to d \in a))) \wedge \forall c \in L\, ((\forall d \in L\, (d \in c \to d \in a)) \to c \in b)$.

Then $\langle L, \in\rangle \models (\forall c\, (c \in b \to \forall d\, (d \in c \to d \in a))) \wedge \forall c\, ((\forall d\, (d \in c \to d \in a)) \to c \in b)$.

That is, $\langle L, \in\rangle \models b = \wp a$.

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2. The *rank* of a set A, $rk(A)$, is defined to be the least $\alpha \in On$ such that $A \subseteq V_\alpha$. Prove that $\forall \alpha \in \mathbf{On}(rk(L_\alpha) = \alpha)$.

This follows from the result proved later on that for all α, $\alpha = \mathbf{On} \cap V_\alpha$.

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3. Let E denote the set of even natural numbers. Prove that $E \in L_{\omega+1}$.

The statement (for ordinals α, β and γ) that $\gamma = \alpha + \beta$, is Σ_0 and so absolute between transitive classes, and so is the statement "α is an ordinal".

Now we can express "n is a natural number" by "n is an ordinal, and n is not a limit ordinal, and for all $m \in n$, m is not a limit ordinal". So this is also Σ_0.

Now we express "$n \in E$" as: "n is a natural number, and either $n = 2.n$, or for some $m \in n$, $n = 2.m$". This is also Σ_0 and so absolute between transitive classes. Let us refer to this statement as $\phi(n)$.

Now L_ω is a transitive classes containing ω as a subset, so $\phi(n)$ is true if and only if $\langle L_\omega, \in\rangle \models \phi(n)$.

Thus $E = \{a \in L_\omega : \langle L_\omega, \in\rangle \models \phi(a)\}$.

So $E \in L_{\omega+1}$.

B.

4. For $\phi(\mathbf{v})$ a formula of LST (without parameters) and a any set, let $\phi_a(\mathbf{v})$ denote the formula (with parameter a) obtained by relativizing $\phi(\mathbf{v})$ to the class a. Prove that for any transitive class A and $a, \mathbf{b} \in A$, $(A, \in) \models \phi_a(\mathbf{b})$ iff $\phi_a(\mathbf{b})$ (ie. $\phi_a(\mathbf{v})$ is A-absolute).

This is because $\phi_x(\mathbf{y})$ is Σ_0, for every quantifier in it is bounded, having the form $\exists z \in x$ or $\forall z \in x$.

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5. Prove that $\forall \alpha, \beta \in \mathbf{On}$, (i) $V_\alpha \cap \mathbf{On} = \alpha$, and (ii) if $\alpha \in V_\beta$, then $V_\alpha \in V_\beta$.

We prove these by induction, having first proved by induction that for all α, V_α is transitive. For, V_0 is empty so trivially transitive. Suppose that V_α is transitive and that $a \in V_{\alpha+1}$. Then $a \subseteq V_\alpha$, and then if $b \in a$, then $b \in V_\alpha$, so by the inductive hypothesis, $b \subseteq V_\alpha$, so $b \in V_{\alpha+1}$. Now if λ is a limit, and $a \in V_\lambda$, then for some $\alpha < \lambda$, $a \in V_\alpha$, so $a \subseteq V_\alpha$ by the inductive hypothesis, so $a \subseteq V_\lambda$.

We now prove (i) and (ii) in parallel.

V_0 is empty, so $V_0 \cap \mathbf{On} = \varnothing = 0$.

The base case for (ii), that if $\beta \in V_0$, then $V_\beta \in V_0$, is vacuous.

Suppose that $V_\alpha \cap \mathbf{On} = \alpha$.

Then $\alpha \subseteq V_\alpha$, so $\alpha \in \wp V_\alpha = V_{\alpha+1}$. But $V_\alpha \subseteq V_{\alpha+1}$, so $\alpha \cup \{\alpha\} \subseteq V_{\alpha+1}$. Now suppose that β is an ordinal and $\beta \in V_{\alpha+1}$. Then $\beta \subseteq V_\alpha$, so by the inductive hypothesis, $\beta \subseteq \alpha$. Hence $\beta \leq \alpha$. So $\mathbf{On} \cap V_{\alpha+1} = \alpha + 1$.

Now suppose that β is an ordinal, and $\beta \in V_{\beta+1}$. Then $\beta \in \alpha + 1$ by the previous paragraph. If $\beta = \alpha$, well $V_\alpha \subseteq V_\alpha$, so $V_\alpha \in \wp V_\alpha = V_{\alpha+1}$. If $\beta < \alpha$, then $\beta \in V_\alpha$, so by the inductive hypothesis, $V_\beta \in V_\alpha$. Now $V_\alpha \subseteq V_{\alpha+1}$, so $V_\beta \in V_{\alpha+1}$.

Now suppose that λ is a limit.

Then $\mathbf{On} \cap V_\lambda = \mathbf{On} \cap \bigcup_{\alpha\in\lambda} V_\alpha = \bigcup_{\alpha\in\lambda} \mathbf{On} \cap V_\alpha = \bigcup_{\alpha\in\lambda} \alpha = \lambda$.

Now if $\beta \in V_\lambda$, then for some $\alpha < \lambda$, $\beta \in V_\alpha$, so $V_\beta \in V_\alpha$ by the inductive hypothesis, so $V_\beta \in V_\lambda$.

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6. A *club* is, by definition, a closed, unbounded class of ordinals. Prove that if U_1 and U_2 are clubs then so is $U_1 \cap U_2$. More generally, suppose that X is a class such that $X \subseteq \omega \times On$. For $i \in \omega$, let $X_i = \{\alpha \in On : \langle i, \alpha \rangle \in X\}$. Suppose that for all $i \in \omega$, X_i is a club. Prove that $\bigcap_{i\in\omega} X_i$ is a club.

The first part follows at once from the second, so we do the second.

We first show that $\bigcap_{i\in\omega} X_i$ is unbounded. For, let α be any ordinal. Let $\alpha_{0,0} = \alpha$. Find ordinals $\alpha_{m,n}$, for $m, n \in \omega$, as follows. Let $\alpha_{m+1,0} = \bigcup_{n\in\omega} \alpha_{m,n}$. Let $\alpha_{m,n+1}$ be the least element of X_n which is $\geq \alpha_{m,n}$.

Now let $\beta = \sup_{m\in\omega} \alpha_{m,0}$. Then since X_n is closed, and $\beta = \sup\{\alpha_{m,n+1} : n \in \omega\}$, $\beta \in X_n$. Thus $\beta \in \cap_{n\in\omega} X_n$.

Now we observe that $\bigcap_{i\in\omega} X_i$ is closed. For suppose that A is a non-empty subset of $\bigcap_{i\in\omega} X_i$. Then for all i, A is a non-empty subset of X_i, so $\sup A \in X_i$. Hence $\sup A \in \bigcap_{i\in\omega} X_i$, as required.

C.

7. (i) It is known that there is a formula $\phi(x)$ of LST (without parameters) such that (in ZF one can prove that) for any set a, $\phi(a)$ iff "$\langle a, \in \rangle \vDash$ ZF and a is transitive". Further, this formula is A-absolute for any transitive class A (see previous sheet). Show that one cannot prove the sentence $\exists x\, \phi(x)$ from ZF. [Hint: Consider the least $\alpha \in \mathbf{On}$ such that $\exists x \in V_\alpha(\phi(x))$.]

Suppose that $\exists x\, \phi(x)$ is provable from ZF.

Then this sentence is true in V.

Then by the Lévy Reflection Principle, there exists α such that $\langle V_\alpha, \in\rangle \models \exists x\, \phi(x)$.

Suppose that α is the least ordinal having this property. Let $a \in V_\alpha$ be such that $\langle V_\alpha, \in\rangle \models \phi(a)$.

Then by absolutenss, since V_α is transitive, $\phi(a)$ is true in V, and a is a transitive set and is a model of ZF.

Hence $\langle a, \in\rangle \models \exists x\, \phi(x)$.

Let $b \in a$ be such that $\langle a, \in\rangle \models \phi(b)$.

Then by absoluteness, and the fact that a is transitive, $\phi(b)$ holds in V.

Now since ϕ is absolute between transitive classes, and all V_γ are transitive, α must be minimal subject to $a \in V_\alpha$, so α is a successor $\beta + 1$, $a \subseteq V_\beta$, and β is least subject to that condition.

Now $b \in a$, so $b \in V_\beta$. But then because ϕ is absolute and V_β is transitive, $\langle V_\beta, \in\rangle \models \phi(b)$, so $\langle V_\beta, \in\rangle \models \exists x\, \phi(x)$, contradicting minimality of α.

(ii) As formulated in the lectures, ZF is a countably infinite collection of axioms (since there is one separation and replacement axiom for each formula of LST, and there are clearly a countably infinite number of such formulas). Prove that there is no finite subcollection, T, say, of ZF, such that $T \vdash$ ZF.

Suppose T is a finite subset of ZF such that $T \vdash$ ZF.

Then $\bigwedge T$ is a single formula from which ZF can be proved.

Now, by the Lévy Reflection Principle, there exists α such that $\langle V_\alpha, \in\rangle \models \bigwedge T$.

But then $\phi(V_\alpha)$ is true, where ϕ is the formula from part (i).

So we can prove $\exists x\, \phi(x)$ from ZF, contradicting part (i).

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8. * What is wrong with the following argument:

Let $\{\sigma_i : i \in \omega\}$ be an enumeration of all the axioms of ZF. By Lévy's Reflection Principle, for each $i \in \omega$, the class $\{\alpha \in On : \langle V_\alpha, \in\rangle \models \sigma_i\}$ (call it X_i) is a club (since $(V, \in) \models \sigma_i$). By question (3) above, $\bigcap_{i\in\omega} X_i$ is a club (we are using question (3) by setting $X = \{\langle i, \alpha\rangle : \alpha \in X_i\}$). In particular, $\bigcap_{i\in\omega} X_i$ is non-empty. Let $\beta \in \bigcap_{i\in\omega} X_i$. Then $\beta \in X_i$ for all $i \in \omega$, so $\langle V_\beta, \in\rangle \models \sigma_i$ for all $i \in \omega$, so $\langle V_\beta, \in\rangle \models$ ZF. Hence $\phi(V_\beta)$ holds, so $\exists x \phi(x)$ (where $\phi(x)$ is the formula in (4)(i)). Since $(V, \in)$ is an arbitrary model of ZF, we have ZF $\vdash \exists x \phi(x)$!

X is not definable in the language of set theory; that is, there is no formula $\phi(x, y)$ which is true exactly when $y \in X_x$.

Thus we cannot prove that the intesection of the X_i is non-empty.

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9. Suppose $F : V \to V$ is a term definable without parameters (ie. the formula defining "$F(x) = y$" has no parameters). Suppose further that it is an *elementary map*, ie. for any formula $\phi(v_0, \ldots, v_{n-1})$ of LST (without parameters), and any $a_0, \ldots, a_{n-1} \in V$,

$$\phi(a_0, \ldots, a_{n-1}) \Leftrightarrow \phi(F(a_0), \ldots, F(a_{n-1})).$$

Prove that F is the identity. [Hint: first show that for all ordinals α, $F(\alpha) = \alpha$, by considering the first β for which $F(\beta) \neq \beta$.]

[Remark: Assuming only ZF, it is not known whether such an elementary map definable *with* parameters can exist other than the identity, although if ZFC is assumed it is known that there is no such.]

Let α be least such that $F(\alpha) \neq \alpha$. Since for all $\beta \in \alpha$, $F(\beta) = \beta$, and since for $\beta \neq \alpha$, elementarity of F tells us that $F(\beta) \neq F(\alpha)$, and since if α is an ordinal, then $F(\alpha)$ must also be an ordinal, we must have that $F(\alpha) > \alpha$.

Let $\phi(x, y)$ express "x and y are ordinals and x is the least ordinal such that $F(x) \neq x$ and $F(x) = y$".

Then $\phi(\alpha, F(\alpha))$ holds.

Now F is elementary, so $\phi(F(\alpha), F(F(\alpha)))$ holds as well.

But this means that $F(\alpha)$ is the least ordinal β satisfying $F(\beta) \neq \beta$, which is false, giving a contradiction.
