Chapter 11

Cohomology, divisors and miracles

§ Pic, CaCl and H^1

def. $\mathcal{O}_X^\times \subset \mathcal{O}_X$ sheaf of invertible functions:

$$\mathcal{O}_X^\times(U) = \{f \in \mathcal{O}_X(U) : \exists g \in \mathcal{O}_X(U) \quad f \cdot g = 1\}$$

(abelian group under multiplication

Thm $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\times)$ as groups.

Proof: want to show:

$$\left\{\begin{array}{c}\text{isom classes of line bdls} \\ \text{that admit a trivialization} \\ \text{under } U_i\end{array}\right\} \longleftrightarrow H^1_{\{U_i\}}(X, \mathcal{O}_X^\times)$$

(and then take colim along $\{U_i\}$).

Take a line bdle $\mathcal{L}$: it's encoded by

$\alpha_{ij} : \mathcal{O}_{U_{ij}} \xrightarrow{\sim} \mathcal{O}_{U_{ij}}$ — isoms of $\mathcal{O}_{U_{ij}}$-modules,

transition maps

so each α_{ij} is multiplication by element $\in \mathcal{O}^\times(U_{ij})$.

Cocycle condition: $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on U_{ijk}.

Rewrite cocycle condition:

$\alpha_{ij} \cdot \alpha_{ik}^{-1} \cdot \alpha_{jk} = 1$, which is the multiplicative form of $s_{ij} - s_{ik} + s_{jk} = 0$.

Get: $\cdot\ (\alpha_{ij}) \in H^1_{\{U_i\}}(X, \mathcal{O}_X^\times)$.

$\cdot\ \mathcal{L}\otimes\mathcal{L}'$ corresponds to $(\alpha_{ij}\cdot\alpha'_{ij})$

Claim: $[(\alpha_{ij})] = [(\tilde{\alpha}_{ij})]$ in $H^1_{\{U_i\}}(X, \mathcal{O}_X^\times)$

iff (α_{ij}) and $(\tilde{\alpha}_{ij})$ give isom. v. B.

In H^1: $[(\alpha_{ij})] = [(\tilde{\alpha}_{ij})]$ iff

$\alpha_{ij} = \beta_j \cdot \tilde{\alpha}_{ij} \cdot \beta_i^{-1}$ for some $\beta_i \in \mathcal{O}_{U_i}^\times$.

(in additive notation: $(s_i) \in C^0 \rightsquigarrow d(s_i) = s_j - s_i$ on U_{ij}).

coboundary

In line bundles:

$$\begin{tikzcd}
\mathcal{O}_{U_{ij}} \arrow[d, "\tilde{\alpha}_{ij}"'] \arrow[rrr, bend left, "\beta_i"] & \tilde{\mathcal{L}}|_{U_{ij}} \arrow[l, "\simeq"'] \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\parallel"] & \mathcal{L}|_{U_{ij}} \arrow[r, "\simeq"] \arrow[d, phantom, "\parallel"] & \mathcal{O}_{U_{ij}} \arrow[d, "\alpha_{ij}"] \\
\mathcal{O}_{U_{ji}} & \tilde{\mathcal{L}}|_{U_{ji}} \arrow[l, "\simeq"'] \arrow[r, phantom, "\simeq"] & \mathcal{L}|_{U_{ji}} \arrow[r, "\sim"] & \mathcal{O}_{U_{ji}}
\end{tikzcd}$$

Taking $\mathcal{L} = \tilde{\mathcal{L}}$ shows that H^1 class does not depend on the choice of trivialization of $\mathcal{L}$, so we are done! :)

Thm. X integral, Noeth, sep. $\Rightarrow$

$$\mathrm{CaCl}(X) \simeq H^1(X, \mathcal{O}_X^\times).$$

Cor: $\mathrm{CaCl}(X) \simeq \mathrm{Pic}(X)$ — as promised! :)

In particular, $D \sim D'$ iff $\mathcal{O}(D) \simeq \mathcal{O}(D')$.

Proof of thm:

consider the exact seq of sheaves

$$0 \to \mathcal{O}^\times \to K^\times \to K^\times / \mathcal{O}^\times \to 0.$$

Take LES:

$$0 \to H^0(X, \mathcal{O}^\times) \to H^0(X, K^\times) \to \underbrace{H^0(X, K^\times/\mathcal{O}^\times)}_{\text{Cartier divisors}} \to$$

$$\to H^1(X, \mathcal{O}_X^\times) \to \underset{\overset{\|}{0}}{H^1(X, K^\times)} \to \dots$$

image is the principal Cartier divisors

Because $K^\times$ is constant and X irreducible

Hence

$$H^0(X, K^\times/\mathcal{O}^\times) \Big/ H^0(X, K^\times) \xrightarrow{\sim} H^1(X, \mathcal{O}_X^\times).$$

Conclusion:

$$H^1(X, \mathcal{O}_X^\times) \underset{\text{always}}{\simeq} \mathrm{Pic}(X) \underset{\substack{X \text{ integral} \\ \text{noeth. sep.}}}{\simeq} \mathrm{CaCl}(X) \underset{\substack{X \text{ also regular} \\ \text{in codim 1}}}{\simeq} \mathrm{Cl}(X)$$

Functoriality of Cl

Prop: 1) $f: X \to Y$ flat $\Rightarrow f^*: \mathrm{Div}(Y) \to \mathrm{Div}(X)$ which factors through Cl

prime divisor $Z \mapsto f^{-1}(Z)$ codim 1 because of flatness

2) $f: X \to Y$ proper $\Rightarrow f_*: \mathrm{Div}(X) \to \mathrm{Div}(Y)$

prime divisor $Z \mapsto \begin{cases} \overline{f(Z)} & \text{if it's a prime divisor} \\ 0 & \text{else} \end{cases}$

($\overline{f(Z)}$ is always closed irred. subscheme, but not always codim 1) which factors through Cl (this uses properness)

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§ Riemann-Roch thm

Recall: D Cartier divisor on X $\rightsquigarrow \mathcal{O}_X(D)$ line Bdl

More generally: D Weil divisor $\rightsquigarrow \mathcal{O}_X(D)$ — an $\mathcal{O}_X$-module

We define $\mathcal{O}_X(D)$:

$$U \mapsto \{0\} \cup \{f \in K \mid \mathrm{div}(f) + D \geq 0 \text{ on } U\}$$

coeffs are positive

it's a condition on allowed zeroes and poles of f

And $\mathcal{O}_X(D)$ is invertible (line Bundle) iff D is locally principal (a Cartier divisor), because: $\exists \{U_i\}$ $\mathcal{O}_X(U_i) \xrightarrow{\sim} \Gamma(U_i, \mathcal{O}_X(D))$

$1 \mapsto f_i \in K$

means exactly that $\tilde{D} = (U_i, f_i)$ is a Cartier divisor and $\mathcal{O}_X(\tilde{D})(U_i) := \mathcal{O}_X(U_i) \cdot \frac{1}{f_i} = \Gamma(U_i, \mathcal{O}_X(\tilde{D}))$; moreover, $\tilde{D} \mapsto D$ under $\mathrm{Cart}(X) \to \mathrm{Div}(X)$.

Riemann-Roch thm

non-examinable page!

C projective smooth algebraic curve $/k=\bar{k}$, $D=\sum n_i[p_i]$ divisor of degree $d=\sum n_i$.

Let $\mathcal{F}:=\mathcal{O}_C(D)$ and $\chi(C,\mathcal{F}):=\sum(-1)^m\cdot\dim_k H^m(C,\mathcal{F})$.

Euler characteristic of $\mathcal{F}$

Then $\chi(C,\mathcal{F}) = \deg D+\chi(C,\mathcal{O}_C) = d+1-g$

$\chi(C,\mathcal{F}) = \dim H^0(C,\mathcal{O}(D))-\dim H^1(C,\mathcal{O}(D))$, $\quad \chi(C,\mathcal{O}_C) = 1-\text{genus}(C)$

$1 = \dim H^0$, $\text{genus}(C) = \dim H^1$ ← usual genus if $k=\mathbb{C}$!

smooth proj alg curves/$\mathbb{C}$ ⟷ compact Riemann surfaces

$X \longmapsto X(\mathbb{C})$

Moral when $k=\mathbb{C}$: for a compact Riemann surface M
linearly independent meromorphic functions with a chosen restriction on the poles depends only on the genus $g(M)$.

Cor. M compact connected Riemann surface, $a\in M$. Then $\exists$ non-constant meromorphic function f on M which has a pole of order $\le g+1$ at a and is holomorphic otherwise.

Proof: $D=(g+1)\cdot[a]$ has deg $g+1$ $\Rightarrow$

$$\dim H^0(M,\mathcal{O}(D)) \overset{R-R}{\geq} d-g+1 = g+1-g+1 = 2,$$

such a function would live here!

and constant functions form a 1-dim subspace.