Gödel Incompleteness Theorems: Solutions to sheet 3

A.

1. Show that consistency is strictly weaker than 1-consistency.

Firstly, any 1*-consistent system* S *is consistent, because there is a formula which* S *does not prove (of the form* $\phi(\overline{n})$*, where* ϕ *is* Σ_0*).*

Now we argue that there is a system that is consistent but 1*-inconsistent.*

Let G *be a* Π_1 *sentence as provided by the First Incompleteness Theorem, such that* G *is neither provable nor disprovable from PA.*

Then $\neg G$ *is* Σ_1 *and not disprovable, so* $\mathrm{PA} \cup \{\neg G\}$ *is consistent.*

Because PA is Σ_1*-complete,* $\neg G$ *must be false.*

Suppose that $\exists x\, \phi(x)$ *is provably equivalent to* $\neg G$ *over PA, so that* $\phi(x)$ *is* Σ_0*.*

Then $\exists x\, \phi(x)$ *is false.*

Hence for all n*,* $\phi(\overline{n})$ *is false, and so* $\neg\phi(\overline{n})$ *is* Σ_0 *and true.*

Since PA is Σ_0*-complete,* $\mathrm{PA} \vdash \neg\phi(\overline{n})$ *for all* n*.*

Thus $\mathrm{PA} \cup \{\neg G\}$ *is consistent, but* 1*-inconsistent.*

2. (i) Show how to construct a sentence, using the Diagonal Lemma, that “says”, “this sentence, when added to PA, results in a system that is ω-inconsistent”.

Use the Diagonal Lemma on the formula in the hint.

B.

3. Show that if a system S is Σ_0-complete and ω-consistent, then it is Σ_2-sound.

Suppose that $S \vdash \exists x\, \forall y\, \phi(x, y)$ *where* ϕ *is* Σ_0*.*

Then there exists n *such that* $S \nvdash \neg\forall y\, \phi(\overline{n}, y)$*; that is,* $S \nvdash \exists y\, \phi(\overline{n}, y)$*.*

Now if S *is* Σ_0*-complete, then it is* Σ_1*-complete. If* $\mathbb{N} \vDash \exists y\, \phi(\overline{n}, y)$*, then* $S \vdash \exists y\, \phi(\overline{n}, y)$*, giving a contradiction. So* $\mathbb{N} \vDash \neg\exists y\, \phi(\overline{n}, y)$*. Hence* $\mathbb{N} \vDash \exists x\, \forall y\, \phi(x, y)$*.*

(i) Prove that the result in the last problem but one is the best possible, in the sense that there exists a system S that is ω-consistent and which proves a false Σ_3-sentence. (Assume that PA is true in $\mathbb{N}$.)

Suppose L *is diagonal with respect to the formula, which we'll write* $H(v_1)$*, in the hint in the last part.*

Then L *is provably equivalent to* $H(\overline{\ulcorner L \urcorner})$*, which is* Σ_3*.*

We now consider the system $\mathrm{PA} \cup \{L\}$*.*

We argue that this system is ω*-consistent.*

For, if it were not, then $H(\overline{\ulcorner L \urcorner})$ *would be true, and so* $\mathrm{PA} \cup \{L\}$ *would be* ω*-inconsistent.*

But then also L *would be true, so* $\mathrm{PA} \cup \{L\}$ *would be true; and any true set of formulae must be* ω*-consistent, and so we have a contradiction.*

Examining the previous two paragraphs, we see that L *must be false, and hence so is* $H(\overline{\ulcorner L \urcorner})$*.*

So $\mathrm{PA} \cup \{L\}$ *is an* ω*-consistent system which proves a false* Σ_3 *sentence.*

4. (i) Show that every finite subset of the axioms of R has a finite model.

Any finite part of R *is true in some* $\mathbb{Z}_n$*, for large enough* n*, where* $\leq$ *is the usual order on the set* $\{0, \ldots, n-1\}$*.*

(ii) Show that R is not finitely axiomatisable.

Obvious from the above.

(iii) Show that Q is a proper extension of R.

There are non-standard structures modelling R but not Q (with total chaos in the non-standard region, since R says nothing at all about the non-standard region but Q at least insists that $\leq$ is a total order).

(iv) Show that PA is a proper extension of Q.

The ordinal ω_1 with ordinal operations satisfies Q but not PA.

C.

5. (i) Show that if a theory S is ω-consistent, then at least one of $S \cup \{X\}$ and $S \cup \{\neg X\}$ is ω-consistent.

Suppose that $S \cup \{X\}$ and $S \cup \{\neg X\}$ are both ω-inconsistent.

Suppose that $S \cup \{X\} \vdash \exists x\, \phi(x)$ and for all n, $S \cup \{X\} \vdash \neg\phi(\overline{n})$, and that $S \cup \{\neg X\} \vdash \exists x\, \psi(x)$ and for all m, $S \cup \{\neg X\} \vdash \neg\psi(\overline{m})$.

So for all n, $S \vdash X \to \neg\phi(\overline{n})$, and for all m, $S \vdash X \to \neg\psi(\overline{m})$. Hence for all n and m, $S \vdash \big(X \to \neg\phi(\overline{n})\big) \wedge \big(\neg X \to \neg\psi(\overline{m})\big)$.

If $(k, l) \mapsto [k, l]$ is the pairing function, define functions $n \mapsto n_1$ and $n \mapsto n_2$ so that for all n, $n = [n_1, n_2]$.

Then for all n, $S \vdash \big(X \to \neg\phi(\overline{n_1})\big) \wedge \big(\neg X \to \neg\psi(\overline{n_2})\big)$.

Now $S \vdash X \vee \neg X$.

So for all n, $S \vdash \big(X \wedge \neg\phi(\overline{n_1})\big) \vee \big(\neg X \wedge \neg\psi(\overline{n_2})\big)$; that is, $S \vdash \neg\big(X \to \phi(\overline{n_1})\big) \vee \neg\big(\neg X \to \psi(\overline{n_2})\big)$, so $S \vdash \neg\Big(\big(X \to \phi(\overline{n_1})\big) \wedge \neg\big(\neg X \to \psi(\overline{n_2})\big)\Big)$.

Also $S \vdash X \to \exists x\, \phi(x)$ and $S \vdash \neg X \to \exists x\, \psi(x)$.

So $S \vdash \exists x \exists y \Big(\big(X \to \phi(x)\big) \wedge \big(\neg X \to \psi(y)\big)\Big)$, so $S \vdash \exists x \Big(\big(X \to \phi(x_1)\big) \wedge \big(\neg X \to \psi(x_2)\big)\Big)$.

Thus S is ω-inconsistent.

(ii) Show that there is one and only one complete ω-consistent extension of PA. Take as given that PA is sound.

If T is an extension with the properties given, then use ω-consistency to eliminate quantifiers, to find that T is true in $\mathbb{N}$ and must therefore be the theory of $\mathbb{N}$.

In slightly more detail, we argue by induction on n that the Σ_n elements of T are precisely the true ones. This is obvious for $n = 0$. If $\exists x\, \phi(x)$ is Σ_{n+1} and belongs to T, then by ω-consistency, some $\phi(\overline{m})$ is not disproved by T and therefore belongs to T by completeness. By the inductive hypothesis, $\phi(\overline{m})$ is true and hence so is $\exists x\, \phi(x)$. Conversely, if $\exists x\, \phi(x)$ is Σ_{n+1} and true, then for some m, $\phi(\overline{m})$ is true, and belongs to T by the inductive hypothesis. By consistency and completeness, $\exists x\, \phi(x)$ belongs to T.

(iii) Explain why the following complete extension S of PA is not ω-consistent. Let $\{X_n : n \in \mathbb{N}\}$ be a listing of all sentences of $\mathscr{L}$. Let K be a sentence such that K is false and PA$\cup\{K\}$ is ω-consistent, and let S_0 be PA$\cup\{K\}$. Let S_{n+1} be $S_n \cup \{X_n\}$ if $S_n \cup \{X_n\}$ is ω-consistent, otherwise let S_{n+1} be $S_n \cup \{\neg X_n\}$. For each i, S_i is ω-consistent by part (i). Let $S = \bigcup_{n \in \mathbb{N}} S_n$.

n-consistency doesn't automatically carry through at limit stages of countable cofinality.