Geometric Group Theory

Problem Sheet 2

1. Let $\langle S|R\rangle$ be a finite presentation of a group G.

i. Explain how to enumerate all words on S representing the identity in G.

ii. Explain how to enumerate all finite presentations of G.

Solution. i) We enumerate all products

$$\prod_{i=1}^{n} x_i r_i^{\pm 1} x_i^{-1}, \;\; r_i \in R,\, x_i \in F(S)$$

in $F(S)$ increasing 'in parallel' n and the lengths of x_i's.

More precisely we do this in steps. In step k we enumerate all such words with $n \leq k$ and $|x_i| \leq k$. This is clearly a finite set of words. Clearly each such word will appear in some step k.

ii) We do several things 'in parallel': We enumerate all possible sequences of Tietze transformations (words on 4 letters) and then we go back and forth along these words applying Tietze transformations using words of $\langle\langle R\rangle\rangle$ and of $F(S)$ according to the transformation.

More formally: We will be writing all presentations using a fixed set of symbols (letters), say $x_1, x_2, x_3, \ldots$.

As noted in part i) if $\langle S|R\rangle$ is a finite presentation we may enumerate all words in $\langle\langle R\rangle\rangle$.

We enumerate now all possible sequences of Tietze moves on a given presentation $\langle S|R\rangle$ as follows: In step n we start by enumerating all words of length n in $T1, T2$ and their inverses (clearly there are finitely many such words). Given such a word if the first letter is $T1$ we enumerate the first n words in $\langle\langle R\rangle\rangle$ and we apply all moves $T1$ corresponding to these words to get n new presentations. If the first letter is $T1^{-1}$ we consider all subsets $R_1 \subset R$ we enumerate the first n elements of $\langle\langle R_1\rangle\rangle$ and if some element of $R - R_1$ appears in this list we apply the corresponding Tietze $T1$ move. If the first letter of the word is $T2$ we enumerate all words of length n in S and for each one of them we apply a $T2$ move obtaining a new presentation. If the first letter is $T2^{-1}$ we check if the relations allow us to eliminate some generator and for each such possible elimination we obtain a new presentation. In this way we obtain a finite set of presentations from the first letter of the T_i-word. Then for each one of them we apply the same procedure to the second letter of the word and so on.

Clearly each presentation of G will appear in some step of this procedure.

2. Let $\langle S|R\rangle$ be a finite presentation of a finite group G. Give an algorithm to solve the word problem for this presentation.

Solution.

The same as the solution of the word problem in the notes for residually finite groups. Finite groups are of course residually finite.

3. Show that if G has a solvable word problem and H is a finitely presented subgroup of G then H also has a solvable word problem.

Solution. Say $H = \langle a_1, ..., a_k | r_1, ..., r_n\rangle$. Let w be a word on $a_1, ..., a_k$. We do two things 'in parallel':

1) We list elements of $<< r_1, ..., r_k >>$ and we check whether w appears in this list

2) we list homomorphisms $f : H \to G$ and we check whether $f(w) \neq 1$.

If $w = 1$ then we will eventually know it by 1). If $w \neq 1$ we will eventually know it by 2).

We remark that it is possible to list homomorphisms $f : H \to G$ as follows. We list k-tuples $h_1, ..., h_k$ of elements of G and we check whether they satisfy the relators $r_1, ..., r_n$. If they do the map $a_i \to h_i$ is a homomorphism. This is possible to check since G has solvable word problem.

4. If H is a finitely generated subgroup of G then the *membership problem* for H asks whether there is an algorithm to decide if $g \in G$ lies in H. Show that the membership problem is solvable for cyclic subgroups of F_n (the free group of rank n). In other words there is an algorithm such that given $u, w \in F_n$ decides whether $u \in< w >$.

Solution. If $w = ava^{-1}$ with v cyclically reduced it is enough to check whether $u = w^n$ for all $n \leq |u|$.

5. Show that the following presentations are presentations of the trivial group:

i) $\langle a, b, c|aba^{-1} = b^2, bcb^{-1} = c^2, cac^{-1} = a^2\rangle$

ii) $\langle a, b|a^n = b^{n+1}, aba = bab\rangle$

iii) $\langle a, b|ab^na^{-1} = b^{n+1}, ba^nb^{-1} = a^{n+1}\rangle$.

Solution.

I am fairly certain the solutions below are not the shortest possible.

i) We note that we have the relations:

$$a^kba^{-k} = b^{2^k},\ b^{-1}ab = ba,\ c^{-1}bc = cb,\ a^{-1}ca = ac$$

We have

$$cac^{-1}bca^{-1}c^{-1} = b^4,\ b^4cb^{-4} = c^{16}$$

so

$$(cac^{-1}bca^{-1}c^{-1})c(cac^{-1}bca^{-1}c^{-1})^{-1} = c^{16} \Rightarrow$$

$$a(c^{-1}b)ca^{-1}(cac^{-1})b^{-1}ca^{-1} = c^{16} \Rightarrow ac(bab^{-1})ca^{-1} = c^{16} \Rightarrow$$

$$ac(bab^{-1})ca^{-1} = c^{16} \Rightarrow acb^{-1}(aca^{-1}) = c^{16} \Rightarrow$$

$$acb^{-1}a^{-1}c = c^{16} \Rightarrow acb^{-1}a^{-1} = c^{15} \Rightarrow (aca^{-1})(ab^{-1}a^{-1}) = c^{15} \Rightarrow$$

$$a^{-1}cb^{-2} = c^{15} \Rightarrow a^{-1}b^{-2}(b^2cb^{-2}) = c^{15} \Rightarrow a^{-1}b^{-2} = c^{11} \Rightarrow b^2a = c^{-11} \Rightarrow ab = c^{-11}$$

Now

$$c^{11}ac^{-11} = a^{2^{11}} \Rightarrow b^{-1}a^{-1}aab = a^{2^{11}} \Rightarrow b^{-1}ab = a^{2^{11}} \Rightarrow ba = a^{2^{11}} \Rightarrow b = a^{2^{11}-1}$$

But then $aba^{-1} = b^2 \Rightarrow b = 1 \Rightarrow c = 1 \Rightarrow a = 1$.

ii) Using $aba = bab$ and induction we get $a^n ba = bab^n$.

$$b^{n+1}ab = b^n bab = b^n aba = ... = aba^{n+1}$$

$$aba^{n+1} = aba^n a = ab^{n+2}a = a^{n+1}ba$$

So

$$b^{n+1}ab = a^{n+1}b = a^{n+1}ba \Rightarrow a = 1 \Rightarrow b^2 = b \Rightarrow b = 1$$

iii) We remark that $ba^{nk}b^{-1} = (ba^n b^{-1})^k = a^{k(n+1)}$. So $b^n a^{n^n} b^{-n} = a^{(n+1)^n}$. Now

$$(ab^n a^{-1})a^{n^n}(ab^n a^{-1})^{-1} = ab^n a^{n^n} b^{-n} a^{-1} = a^{(n+1)^n}$$

and

$$(ab^n a^{-1})a^{n^n}(ab^n a^{-1})^{-1} = b^{n+1}a^{n^n}b^{-(n+1)} = ba^{(n+1)^n}b^{-1}$$

Therefore

$$ba^{(n+1)^n}b^{-1} = a^{(n+1)^n} \Rightarrow ba^{n(n+1)^n}b^{-1} = a^{n(n+1)^n} \Rightarrow a^{(n+1)^{n+1}} = a^{n(n+1)^n}$$

It follows that

$$a^{(n+1)^n} = 1$$

$$ba^n b^{-1} = a^{n+1} \Rightarrow (ba^n b^{-1})^{(n+1)^{n-1}} = a^{(n+1)^n} \Rightarrow a^{n(n+1)^{n-1}} = 1$$

It follows that

$$a^{n(n+1)^{n-1}} = a^{(n+1)^n} \Rightarrow a^{(n+1)^{n-1}} = 1$$

We continue inductively and we get

$$a^{n+1} = 1 \Rightarrow a^n = 1 \Rightarrow a = 1 \Rightarrow b = 1$$

6. An infinite finitely generated group is called almost finite if all its quotients are finite groups. Show that every infinite finitely generated group has a quotient that is almost finite.

Solution Let $G =< S|R >$ be a given f.g. infinite group. We enumerate all words on S and we go through the list asking whether adding w_i to relations R results to a finite group. If it does not we add it, if not we go to the next word. If this stops we get the quotient we need. If it goes on forever we add all these countably many relators. We remark that the group we obtain is infinite. This is because finite groups are finitely presented so if it were finite we would have already all the relators in a finite stage. But now we can not add more relators so the quotient we have has the required property.

Alternatively order normal subgroups N such that G/N is infinite by inclusion. An ascending union of such subgroups has the same property since if G/N is finite it is finitely presented so we can find all its relations in a fixed subgroup in the union. So by Zorn's lemma G/N is infinite. On the other hand any quotient of it is finite.

7. i. Show that G is residually finite if and only if for every $g \in G$ there is some finite index subgroup H of G, such that $g \notin H$.

ii. Show that if G has a finite index subgroup which is residually finite then G itself is residually finite.

Solution.

i. Clearly if G is r.f. this holds. Convesely if $g \notin H$ with H f.i. then there is a normal subgroup $N \subseteq H$ of finite index. Then $f : G \to G/N$ satisfies $f(g) \neq 1$, so G is r.f.

ii. We remark that by part i G is residually finite if and only if for every $g \in G$ there is a finite index subgroup H of G st $g \notin H$. Let K be a finite index res. finite subgroup of G. Take $g \in G$. If $g \notin K$ we are done. Otherwise there is a finite index subgroup of K, H such that $g \notin H$. But H is f.i. in G.

8. Let G be a residually finite group. Show that if G has finitely many conjugacy classes of elements of finite order then G has a torsion free finite index subgroup.

Solution Let $g_1, ..., g_n$ be representatives of these conjugacy classes. Take $f : G \to A$, A finite, such that $f(g_i) \neq 1$ for all i. Then $ker\, f$ is a torsion free finite index subgroup of G.

9. Give an example of a residually finite group which is not Hopf.

Solution An infinite direct sum of Z's or a free group of infinite rank will do.

10. If H is a subgroup of the free group F_n of index $|F_n : H| = r$ show that H is a free group of rank $r(n-1)+1$. (*hint:* look closely at the proof that H is free).

Solution H acts on the Cayley graph, T, of F_n with r orbits of vertices. Let X be a subtree of T intersecting each orbit at exactly 1 vertex. Then X has r vertices, so it has $r-1$ edges. We count how many (geometric) edges are are adjacent to X (that is have one vertex on X): Since we have r vertices and $2n$ edges leave from each vertex we have $2rn$ edges leaving from these vertices. However $r-1$ lie in X so these are counted twice. Since we want to count only edges leaving from X we subtract $r-1$ edges so we have

$$2rn - (r-1) - (r-1) = 2(r(n-1)+1)$$

Recall now that if we collapse all translates of X to points we obtain the Cayley graph of H with respect to a free basis. We remark that the number of edges adjacent to each vertex is equal to the number of edges adjacent to X in T. Note that the cardinality of the free basis is $\frac{1}{2}$ of the number of edges leaving a vertex in the Cayley graph.

So the rank of H is $r(n-1)+1$.

11. If $g \neq 1$ is an element of F_n show that the normalizer of $< g >$ in F_n is a cyclic group.

Solution If u is an element of the normalizer $ugu^{-1} = g^{\pm 1}$. However the group $< u, g >$ is free. If it is free of rank 2 then $\{u, g\}$ is a basis since it is a generating set. But then $ugug^{\pm 1} \neq 1$ since it is a reduced word. So $< u, g >$ is cyclic, therefore $ugu^{-1} = g$. If the normalizer is not cyclic then it is free with basis which has at least 2 elements a, b. But then either aga^{-1} or bgb^{-1} is not equal to g (as it is a word that starts with a different letter than g), a contradiction. So the normalizer of $< g >$ is a cyclic group.

12. Show that every cyclic subgroup of F_n (the free group of rank n) is separable.

Solution Enough to do for $n = 2$. Let $v \in F_2$ and let $w \notin < v >$. Certainly we can find a homomorphism to $Symm(X)$ as in the notes so that $f(w) \neq f(v)$. The issue is to make sure that $f(v^n) \neq f(w)$ for any n. We may assume v is cyclically reduced (otherwise just replace v by a cyclically reduced conjugate of it gvg^{-1}-and replace w by gwg^{-1} as well). Now consider k such that $N = |v|^k > |w|$ and let X be the set of reduced words of length N. We define as in the notes maps $\alpha, \beta \in Sym(X)$ acting as the generators a, b on reduced words of length $\leq N-1$. In fact slightly more generally we define $\alpha(g) = ag$ for all g in X such that $ag \in X$- and similarly for β.

Now we identify the elements v^k and v^{-k} in X. We need to check that this is possible as if some permutation say α is already defined on these two elements and it is defined in different ways then this identification is not possible.

Note that if $v^k = a_1 ... a_r$ ($a_i \in \{a^{\pm 1}, b^{\pm 1}\}$) then the permutation corresponding to the letter a_1^{-1} is defined already on v^k. Similarly the permutation corresponding to a_r is already defined on $v^{-k} = a_r^{-1}(a_{r-1}^{-1} ... a_1^{-1})$. Since v is cyclically reduced $a_1^{-1} \neq a_r$ so it is possible to identify v^k and v^{-k}. Note that after this identification the permutations corresponding to a_1^{-1}, a_r are both defined on the new point.

It follows that the maps α, β are still well defined after this identification. Finally we extend α, β to the rest of X in any way. Then $v^n \cdot e = v^r$ where $r \equiv n \, mod \, 2k$, $r \in [-k, k]$, so $v^r \cdot e \neq w \cdot e$, therefore $f(v^n) \neq f(w)$ for any n.

13. Determine the center of the group $\langle a, b | a^2 = b^3 \rangle$.

Solution This group is an amalgam of $< a >, < b >$ over $< a^2 = b^3 >$. So the center is contained in $< a^2 >$ and we see that it is in fact equal to it.

14. Show that a finite group H acting on a tree T either fixes a vertex of T or fixes a geometric edge of T (ie $H \cdot e \subset \{e, \bar{e}\}$ for some edge e). Deduce that any finite subgroup of an amalgam $A *_C B$ is contained in a conjugate of A or B.

Solution Consider the smallest subtree X of T containing the H-orbit of a given vertex v. We remark that X is H-invariant since $hX \cap X$ is a tree containing Hv for all $h \in H$. To see this note that $hX \cap X$ is a tree as the intersection of two trees is a tree. It also contains Hv so it is equal to X.

If $X = v$ we are done. Otherwise erase all terminal edges of X and remark that the tree you get in this way is again H-invariant by definition. Continue the same way and you end up either with a vertex fixed by H or by a geometric edge fixed by H.

The amalgam $G = A *_C B$ acts on a tree T with stabilizers of vertices conjugates of A, B. So a finite subgroup of G fixes a vertex of T since the action is without inversions. It follows that it is contained in a conjugate of A or B.

15. Show that if A, B are residually finite then $A * B$ is also residually finite.

Solution If $w = c_1 ... c_n$ is a reduced word in $A * B$ define homomorphisms $f : A \to A_1$, $g : B \to B_1$ (A_1, B_1 finite) such that if $c_i \in A, f(c_i) \neq 1$ and if $c_i \in B$ $g(c_i) \neq 1$. By the universal property of the amalgam there is a homomorphism $F : A * B \to A_1 * B_1$ such that F restricted to A is f and restricted to B is g. So $F(w) \neq 1$. We remark now that $A_1 * B_1$ has a finite index free subgroup so $A_1 * B_1$ is residually finite. So there is a fomomorphism $G : A_1 * B_1 \to C$, C finite, such that $G(F(w)) \neq 1$.