Geometric Group Theory

Problem Sheet 4

We use the notation from Lecture Notes, $X \sim Y$, for two metric spaces that are quasi-isometric.

1. i) Show that the relation of quasi-isometry of metric spaces $\sim$ is an equivalence relation.

ii) Let S_1, S_2 be finite generating sets of a group G. Show that $\Gamma(S_1, G) \sim \Gamma(S_2, G)$.

Solution. i) Let $f : X \to Y$ a (K, A)-quasi-isometry. Define a 'quasi-inverse' $g : Y \to X$ as follows: Given $y \in Y$ pick $x \in X$ such that $d(y, f(x)) \leq A$. Define $g(y) = x$. Then g is also a quasi-isometry: Let $x \in X$ and $y = f(x)$ then $g(y) = x_1$ for some x_1 for which $d(f(x), f(x_1)) \leq A$. So $d(x, x_1) \leq KA + A$.

ii) We consider the identity map on the vertices $f : \Gamma(G, S_1) \to \Gamma(G, S_2)$. We can write each element of S_1 as a word on S_2 and each element of S_2 as a word on S_1. The maximum length of all these words controls the quasi-isometry constants.

2. *Given $\epsilon, \delta > 0$ a subset N of a metric space X is called an (ϵ, δ)-*net *(or simply a* net*) if for every $x \in X$ there is some $n \in N$ such that $d(x, n) \leq \epsilon$ and for every $n_1, n_2 \in N$, $d(n_1, n_2) \geq \delta$.*

A set N that satisfies only the second condition (i.e. for every $n_1, n_2 \in N$, $d(n_1, n_2) \geq \delta$) is called δ-separated.

i) Show that any metric space X has a $(1, 1)$-net.

ii) Show that if $N \subset X$ is a net then $X \sim N$.

iii) Show that $X \sim Y$ if and only if there are nets $N_1 \subset X, N_2 \subset Y$ and a bilipschitz map $f : N_1 \to N_2$.

iv) Let G be a f.g. group. Show that $H < G$ is a net in G if and only if H is a finite index subgroup of G.

Solution. i) Let N be a maximal subset of X such that for any $a, b \in N$ $d(a, b) \geq 1$. Such an N exists by Zorn's lemma. Now if $x \in X$ and $d(x, a) \geq 1$ for any $a \in N$ then N is not maximal. So there is some $a \in N$ such that $d(a, x) \leq 1$.

ii) The inclusion $N \to X$ is a quasi-isometry.

iii) Let $f : X \to Y$ be a (K, A)-quasi-isometry. Pick N_1 an (n, n)-net in X with $n = 2K(A + 1) + A$ (sufficiently large). Then $d(f(x), f(y)) \geq 1$ for $x \neq y$ so f is injective on N_1. Also

$$d(f(x), f(y)) \leq Kd(x, y) + A \leq KAd(x, y)$$

and

$$d(f(x), f(y)) \geq \frac{d(x,y)}{K} - A \geq \frac{d(x,y)}{K} - \frac{d(x,y)}{2K} \geq \frac{d(x,y)}{2K}$$

Finally since for any $y \in Y$ there is an $x \in X$ such that $d(y, f(x)) \leq A$ and there is an $a \in N_1$ with $d(a,x) \leq n$ we have that

$$d(f(n), y) \leq A + Kn + A.$$

so $f(N_1) = N_2$ is a net in Y.

iv) Clearly if H is of index n then H is an $(n, 1)$ net in G. Assume that H is an $(n, 1)$ net in G. Let's say that there are M words on the generating set of G of length $\leq n$. For every $g \in G$ $gw \in H$ for some word of length $\leq n$. So $g \in Hw^{-1}$. It follows that the index of H in G is bounded by M.

3. Prove that for every $K \geq 1$ and $A \geq 0$ there exists $\lambda \geq 1$, $\mu \geq 0$ and $D \geq 0$ such that the following is true. Given a (K, A)-quasi-geodesic $q : I \to X$ of endpoints x, y in a geodesic metric space X there exists a (continuous) path $\alpha : I' \to X$ of endpoints x, y such that:

1. for all $t, s \in I$,

$$\textit{length}\,(\alpha([t,s])) \leq \lambda d(\alpha(t), \alpha(s)) + \mu;$$

2. for every $x \in I$, $d(q(x), \alpha(I')) \leq D$;
3. for every $t \in I'$, $d(\alpha(t), q(I)) \leq D$.

Solution. Let $t_0, t_1, ..., t_n$ be points in the interval I such that t_0, t_n are its endpoints, $|t_{i+1} - t_i| = 1$ for all $0 \leq i \leq n-1$, $|t_{n+1} - t_n| \leq 1$. Consider α to be the polygonal line with geodesic edges $[x, q(t_1)] \cup [q(t_1), q(t_2)] \cup \cdots \cup [q(t_{n-1}), y]$, parametrized by its arc length.

The last two conditions are satisfied with $D = \frac{K}{2} + A$ and the first with $\lambda = K^2$ and $\mu = K(A+1) + 2A$.

4. Let X be a δ-hyperbolic geodesic metric space. If L is a geodesic in X and $a \in X$ we say that $b \in L$ is a projection of a to L if

$$d(a, b) = \inf\{d(a, x) : x \in L\}.$$

Show that if b_1, b_2 are projections of a to L then $d(b_1, b_2) \leq 2\delta$.

Solution. This follows easily by considering the geodesic triangle $[a, b_1, b_2]$.

5. Let X be a geodesic metric space.

If $\Delta = [x, y, z]$ is a geodesic triangle in X, then there is a metric tree (a 'tripod' if Δ is not degenerate) T_Δ with vertices x', y', z' (the endpoints when T_Δ is not a segment) such that there is an onto map $f_\Delta : \Delta \to T_\Delta$ that restricts to an isometry from each side $[x, y], [y, z], [x, z]$ to the corresponding segments $[x', y'], [y', z'], [x', z']$ in the tree. We denote by c_Δ the point $[x', y'] \cap [y', z'] \cap [x', z']$ of T_Δ.

We say that a geodesic triangle $\Delta = [x, y, z]$ in a geodesic metric space is δ-*thin* if for every $t \in T_\Delta = [x', y', z']$, $diam(f_\Delta^{-1}(t)) \leq \delta$.

Prove that the following are equivalent:

1. There is a $\delta \geq 0$ such that all geodesic triangles in X are δ-slim.
2. There is a $\delta' \geq 0$ such that all geodesic triangles in X are δ'-thin.

Solution. This appears as Theorem 6.4, with proof, in the Lecture Notes. Please make sure that in class the students understand the two definitions and their equivalence.

6. Let $G = \langle S \rangle$ be δ-hyperbolic for some $\delta \in \mathbb{N}$, $\delta \geq 1$.

1. Assume that for some $g \in G, x \in \Gamma(S, G)$ with $d(x, gx) > 100\delta$ we have that $d(x, g^2x) \geq 2d(x, gx) - 12\delta$.

   Prove that
$$d(x, g^n x) \geq nd(x, gx) - 16n\delta$$
   for all $n \in \mathbb{N}$.

2. Assume that g is an element of infinite order in G. Prove that there are constants $c > 0, d \geq 0$ such that
$$d(1, g^n) \geq cn - d$$
   for all $n \in \mathbb{N}$.

3. Show that G has no subgroup isomorphic to $< x, t | txt^{-1} = x^2 >$.

Solution. 1. This is Lemma 6.4 in the Lecture Notes.
2. This is Proposition 6.4 in the Lecture Notes.
3. $t^n x t^{-n} = x^{2^n}$ which contradicts the fact that x^n is a quasi-geodesic.

7. Let $G =< S|R >$ be a Dehn presentation of a of a δ-hyperbolic group. Show that we can decide whether a word w on S represents an infinite order element.

Solution. To clarify, our input for the algorithm is the finite presentation $< S|R >$ and δ.

1st solution: We use a Dehn presentation and using the solution to the conjugacy problem we check successively for the powers of w, w^k, whether they are conjugate to an element of length $\leq \max\{|r| + 2\}$ where r ranges over all relations of the Dehn presentation. Eventually we will either find that $w^k = 1$ or we will find two powers w^k, w^m which are conjugate to the same element a. It follows that these are conjugate so there is some t such that $tw^k t^{-1} = w^m$. However this contradicts the fact that $< w >$ is a quasi-geodesic as in exercise 8. So either some power is equal to 1 or some power is not conjugate to any element of length $\leq \max\{|r| + 2\}$ (and hence w is of infinite order).

2nd solution: Enumerate powers w^n and check if they are equal to 1. In parallel try to find a vertex m of the Cayley graph and a power w^k such that $d(w^{2k}m, w^k m) > 2d(m, w^k m) - 12\delta$ and $d(e, w^k) > 100\delta$. If w is of finite order the first procedure will terminate. If w is of infinite order then by the proof of the proposition 6.4 in the notes showing that $< w >$ is a quasi-geodesic w^k and m with the above properties exist and we can detect them since the word problem is solvable in G.

8. Let $G =< S|R >$ be a Dehn presentation of a δ-hyperbolic group. Show that we can decide whether a word w on S lies in the subgroup $< v >$.

Solution. To clarify, our input for the algorithm is the finite presentation $< S|R >$, δ and the words v, w.

The proof of proposition 6.4 shows that there is some vertex m in the Cayley graph and some power v^k such that $d(v^{2k}m, v^k m) \geq 2d(v^k m, m) - 12\delta$. However since we can solve the word problem we can find v^k, m just by calculating multiplication tables for larger and larger balls and powers of v. Once those are found we get an estimate, as in proposition 6.4, of the form $d(v^n, e) \geq cn - d$ for some $c, d > 0$. So it is enough to check whether $c^n = w$ for all n for which $cn - d \leq |w|$.