C8.2 Stochastic Analysis and PDEs

Sheet 3 — HT23

For Tutors Only — Not For Distribution

Section A

1. Let Y be a Poisson process with parameter λ and define

$$X_n(t) = \frac{1}{n}\big(Y(n^2 t) - \lambda n^2 t\big).$$

Find A_n, the infinitesimal generator for X_n, and identify the limit of the sequence $(A_n)_n$ as $n \to \infty$ and the corresponding stochastic process.

Solution: Consider $f \in C^2(\mathbb{R})$. We saw in a previous problem sheet (and in the lecture notes) that

$$\begin{aligned}
A_n f(x) &= \lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}^x[f(X_n(t)) - f(x)] \\
&= \lim_{t\downarrow 0} \frac{1}{t}\left\{ n^2\lambda t f(x + \frac{1}{n} - n\lambda t) + (1 - n^2\lambda t) f(x - n\lambda t) - f(x) + \mathcal{O}(t^2) \right\} \\
&= n^2\lambda\big(f(x + \frac{1}{n}) - f(x)\big) + \lim_{t\downarrow 0}\frac{1}{t}\left\{\big(f(x - \lambda n t) - f(x)\big) + \mathcal{O}(t)\right\} \\
&= n^2\lambda\left(f(x+\frac{1}{n}) - f(x) - \frac{1}{n}f'(x)\right). \qquad (1)
\end{aligned}$$

Using the Taylor expansion $f(x + \frac{1}{n}) = f(x) + n^{-1}f'(x) + \frac{1}{2}n^{-2}f''(x) + o(n^{-2})$, we see that

$$\lim_{n\to\infty} A_n f(x) = \frac{\lambda}{2} f''(x)\ ,$$

which is the generator of Brownian motion with volatility λ, i.e. constant time change of Brownian motion. (This convergence happens locally uniformly in x - if we suppose more control on f, such as a uniformly bounded third derivative, then this happens globally in x, i.e. $\sup_x |A_n f(x) - \lambda f''(x)| \to 0$ as $n \to \infty$. This is helpful in connection with Theorem 2.38 in the lecture notes which allows one to pass from convergence of generators to convergence of semigroups.)

2. In proving that sequences of Markov chains converge to diffusions, we have to verify three conditions on the jumps of the chain. Let us write ΔX^h for the increment of the hth chain over a single jump (in the discrete case in which the time between jumps is h) or over an infinitesimal time interval of length h (in the continuous case). Our conditions amount to checking that $\mathbb{E}[\Delta X^h]/h$ and $\mathbb{E}[(\Delta X^h)^2]/h$ both converge as $h \to 0$ and that $\mathbb{P}[|\Delta X^h| > \epsilon]/h \to 0$ as $h \to 0$. Prove that this last condition is implied by the (often more convenient) condition $\mathbb{E}[(\Delta X^h)^4]/h \to 0$.

   Solution: This is clear:

$$\frac{1}{h}\mathbb{P}[|\Delta X^h| > \epsilon] = \frac{1}{h}\mathbb{P}[(\Delta X^h)^4 > \epsilon^4] \leq \frac{1}{\epsilon^4}\frac{1}{h}\mathbb{E}[(\Delta X^h)^4],$$

   by Markov's inequality.

Section C

3. Suppose that X is a *Feller branching process*, that is $X \geq 0$ solves the s.d.e.

$$dX_t = aX_t dt + \sqrt{\gamma X_t} dB_t,$$

for suitable constants $a \in (-\infty, \infty)$ and $\gamma > 0$. By considering the duality function $F(X,Y) = \exp(-XY)$, show that X has a *deterministic* dual and use it to establish $\mathbb{P}[X_t \neq 0]$.

Solution: Apply Itô's formula to find

$$de^{-yX_t} = -ye^{-yX_t} dX_t + y^2 e^{-yX_t} \gamma X_t dt = (-ay + \gamma y^2) X_t e^{-yX_t} dt + \text{ martingale.}$$

If we choose Y_t to solve

$$\frac{dY_t}{dt} = aY_t - \gamma Y_t^2, \tag{2}$$

where to guarantee integrability we take $Y_0 \geq 0$, then we have

$$\mathbb{E}[e^{-X_t Y_0}] = \mathbb{E}[e^{-X_0 Y_t}].$$

Thus this defines a deterministic dual.

Note that $\mathbb{P}[X_t = 0] = \lim_{\theta \to \infty} \mathbb{E}[\exp(-\theta X_t)]$ and so we solve (2) with $Y_0 = \theta$. The equation is separable:

$$\left(\frac{1}{Y} + \frac{\gamma}{a - \gamma Y} \right) dY = adt,$$

and so

$$\frac{Y}{a - \gamma Y} \frac{a - \gamma\theta}{\theta} = e^{at}$$

and rearranging

$$Y(t) = \frac{a\theta e^{at}}{a - \gamma\theta + \gamma\theta e^{at}}.$$

Thus

$$\mathbb{P}[X_t \neq 0] = 1 - \lim_{\theta \to \infty} \exp\left(-\frac{a\theta e^{at}}{a - \gamma\theta + \gamma\theta e^{at}} X_0 \right) = 1 - \exp\left(\frac{a}{\gamma} \frac{1}{1 - e^{at}} X_0 \right).$$

4. Consider the *Wright-Fisher diffusion with selection*, which has generator

$$Af = \frac{1}{2}x(1-x)f''(x) + sx(1-x)f'(x),$$

for suitable $\mathcal{C}^2$ functions f on $[0,1]$ where s is a constant (called the *selection coefficient*). Use duality to check that the martingale problem has a unique solution.

Solution: Without loss of generality, $s < 0$ (otherwise consider $1 - X$). Look for a moment dual:

$$dX^n = nX^{n-1}dX + \binom{n}{2}X^{n-2}X(1-X)dt = -ns\big(X^{n+1}-X^n\big)dt + \binom{n}{2}\big(X^{n-1}-X^n\big)dt + \text{ martingale.}$$

Then we see that there is a moment dual which is a birth and death process with rates:

$$N_t \mapsto \begin{cases} N_t + 1 & \text{at rate } sN_t, \\ N_t - 1 & \text{at rate } \binom{N_t}{2}, \end{cases}$$

and uniqueness of the solution to the martingale problem follows by the usual arguments.

5. The Ornstein-Uhlenbeck process on $\mathbb{R}$ is the one-dimensional diffusion with generator

$$Af(x) = \frac{1}{2}f''(x) - xf'(x).$$

Prove that both ∞ and $-\infty$ are natural boundaries.

Solution: The speed and scale are determined by

$$S'(\xi) = e^{\xi^2}, \qquad m(\xi) = e^{-\xi^2}.$$

To ascertain the nature of the boundary at infinity, we calculate

$$\int_x^\infty \int_x^\xi dM(\eta)dS(\xi) \quad \text{and} \quad \int_x^\infty \int_x^\eta dS(\xi)dM(\eta).$$

The first of these is easily seen to be infinite. For the second, note that

$$\int_x^\infty \int_x^\eta e^{\xi^2} d\xi e^{-\eta^2} d\eta = \int_x^\infty \int_\eta^\infty e^{-\xi^2} d\xi e^{\eta^2} d\eta.$$

Using integration by parts (twice), we obtain

$$\begin{aligned}\int_\eta^\infty e^{-\xi^2} d\xi = \frac{1}{2}\int_\eta^\infty \frac{1}{\xi} 2\xi e^{-\xi^2} d\xi = \frac{1}{2\eta}e^{-\eta^2} - \frac{1}{2}\int_\eta^\infty \frac{e^{-\xi^2}}{\xi^2} d\xi \\ = \frac{1}{2\eta}e^{-\eta^2} - \frac{1}{4\eta^3}e^{-\eta^2} + \frac{3}{4}\int_\eta^\infty \frac{e^{-\xi^2}}{\xi^4} d\xi,\end{aligned}$$

and therefore

$$\int_\eta^\infty e^{-\xi^2} d\xi \approx \frac{1}{2\eta}e^{-\eta^2}, \qquad \text{as } \eta \to \infty.$$

By choosing x_0 large enough we see that $\int_x^\infty \int_x^\eta dS(\xi)dM(\eta)$ is infinite, as required.

6. A Galton Watson branching process is a discrete time Markov chain, $\{Z_n\}_{n\geq 1}$, which is often used to model the growth of a population. The evolution is simple. Each individual leaves behind a random number of offspring in the following generation, according to some distribution, independently of all other individuals. Suppose that the mean number of offspring of each individual is a, the variance is σ^2 and, say, the third moment is bounded. Write Z_0 for the initial population size.

   1. What is the expected population size after N generations?
   2. If we are modelling a very large population, whose size at time zero is NX_0 for some large N and $a \approx 1 + \frac{\mu}{N}$, then find a diffusion approximation for the population size at time Nt in units of size N.

Solution: The expected population size is $Z_0 a^N$ (e.g. by recursion).

Let

$$X_t = \frac{Z_{\lfloor Nt \rfloor}}{N}$$

and consider the change ΔX in one generation (a time interval of length $1/N$).

$$\mathbb{E}[\Delta X] = \frac{1}{N} NX\big(1 + \frac{\mu}{N}\big) - X = \mu X \frac{1}{N}$$

and

$$\mathbb{E}[(\Delta X)^2] = \frac{1}{N^2}(NX)\sigma^2 + \mathcal{O}\big(\frac{1}{N^2}\big) = \sigma^2 X \frac{1}{N} + \mathcal{O}\big(\frac{1}{N^2}\big).$$

To check that we can take a diffusion approximation, note that for the rescaled population to make a jump of size ϵ in a single generation requires the sum of Z independent random variables to deviate from their mean by at least $\epsilon Z/2$ and the finite third moment condition guarantees that we have a Central Limit Theorem, so the probability of this decays exponentially fast in Z which is order N.

The diffusion approximation is a continuous state branching process, which is for example a weak solution to the s.d.e.

$$dX_t = \mu X_t dt + \sigma \sqrt{X_t} dB_t.$$