C8.2 Stochastic Analysis and PDEs

Sheet 4 — HT23

For Tutors Only — Not For Distribution

Section A

1. Let $U \subset \mathbb{R}^d$ be a connected and open set, and $u : U \to \mathbb{R}$ measurable and locally bounded (for every $x \in U$ there exist an open set A such that $x \in A$ and $u(A)$ is a bounded set). Show that the following are equivalent

   1. $\Delta u(x) = 0$ for all $x \in U$,
   2. for any $x \in U$, $r > 0$, and ball $B(x, r) \subset U$,
   $$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)dy$$
   where $|\cdot|$ denotes the Lebesgue measure.
   3. for any $x \in U$, $r > 0$, and ball $B(x, r) \subset U$
   $$u(x) = \frac{1}{\sigma_{x,r}(\partial B(x,r))} \int_{\partial B(x,r)} u(y)dy$$
   where $\sigma_{x,r}$ is the surface measure on $\partial B(x, r)$.

   Solution: Can be found in most analysis books, see for example Theorem 3.2 in Moerters&Peres book "Brownian Motion". The equality 2-3 is less important but all students should know that 1-2 are equivalent.

2. Let $d \geq 3$ and $U \subset \mathbb{R}^d$ the unit disc and let T be the first exit time from U. Show that
$$\int_{\partial U} \frac{1 - |x|^2}{|x - z|^d} G(z, y)\, \pi(dz) = c(d) \frac{|y|^{d-2}}{|x|y|^2 - y|^{d-2}} \text{ for all } x, y \in U$$
where $c(d) = \Gamma(d/2 - 1)/(2\pi^{d/2})$.

   Solution: In Lecture notes, Lemma 7.22 I had to skip the proof during the lecture.

3. Show that
$$\int_0^\infty p(t, x, y)dt = \frac{\Gamma(\frac{d}{2} - 1)}{2\pi^{d/2}} |x - y|^{2-d}$$
where $p(t, x, y) = (2\pi t)^{-d/2} e^{-\frac{|x-y|^2}{2t}}$

Solution: Direct calculation by switiching to polar coordinates; see e.g. proof of Theorem 7.19 in lecture notes.

Section B

4. We can use the Feynman-Kac representation to find the partial differential equation solved by the transition densities of solutions to stochastic differential equations. Suppose that

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{1}$$

For any set B let

$$p_B(t, x; T) \triangleq \mathbb{P}\left[X_T \in B \middle| X_t = x\right] = \mathbb{E}\left[\mathbf{1}_B(X_T) \middle| X_t = x\right].$$

Use the Feynman-Kac representation (assuming integrability conditions are satisfied) to write down an equation for

$$\frac{\partial p_B}{\partial t}(t, x; T)$$

By letting $B \to \{y\}$ (e.g. a ball of radius ϵ around y) deduce that the transition density $p(t, x; T, y)$ ("the probability of being at time t in x and at time T in y") of the solution $(X_s)_{s \geq 0}$ to the stochastic differential equation (1) solves

$$\begin{aligned} \frac{\partial p}{\partial t}(t, x; T, y) + Ap(t, x; T, y) &= 0 \\ p(t, x; T, y) &\to \delta_y(x) \quad \text{as } t \to T, \end{aligned} \tag{2}$$

where A is the infinitesimal generator of X. Equation (2) is known as the *Kolmogorov backward equation* since it operates on the "backward in time" variables (t, x).

[*You can assume that the transition density exists and that the above PDE is well-posed; in particular you do not need to give a rigorous definition of the Dirac delta* δ_y].

Solution: By the Feynman-Kac representation (subject to the integrability condition)

$$\begin{aligned} \frac{\partial p_B}{\partial t}(t, x; T) + Ap_B(t, x; T) &= 0 \\ p_B(T, x; T) &= \mathbf{1}_B(x), \end{aligned} \tag{3}$$

where

$$Af(t, x) = \mu(t, x)\frac{\partial f}{\partial x}(t, x) + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 f}{\partial x^2}(t, x).$$

Writing $|B|$ for the Lebesgue measure of the set B, the transition density of the process $(X_s)_{s \geq 0}$ is given by

$$p(t, x; T, y) \triangleq \lim_{B \to y} \frac{1}{|B|}\mathbb{P}\left[X_T \in B \middle| X_t = x\right].$$

(We are assuming existence of the density). Since the equation (**??**) is linear, we have proved that the transition density of the solution $(X_s)_{s \geq 0}$ to the stochastic differential equation (1) solves (2) as required.

5. We continue the above Question: our aim is to obtain an equation acting on the *forward* variables (T, y). By using integration by parts, show that

$$\frac{\partial p}{\partial T}(t,x;T,y) = A^* p(t,x;T,y) \tag{4}$$

where

$$A^* f(T,y) = -\frac{\partial}{\partial y}\left(\mu(T,y)f(T,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t,Y)f(T,y)\right).$$

Equation (4) is the *Kolmogorov forward equation* of the process $(X_s)_{s\geq 0}$.

[*Hint: State the Chapman-Kolmogorov formula in terms of* p *and differentiate under the integration sign.*]

Solution: By the Markov property of the process $\{X_t\}_{t\geq 0}$, for any $T > r > t$

$$p(t,x;T,y) = \int p(t,x;r,z)p(r,z;T,y)dz.$$

Differentiating with respect to r and using (2),

$$\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial r}p(t,x;r,z)p(r,z;T,y) - p(t,x;r,z)Ap(r,z;T,y)\right\}dz = 0.$$

Now integrate the second term by parts to obtain

$$\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial r}p(t,x;r,z) - A^* p(t,x;r,z)\right\}p(r,z;T,y)dz = 0.$$

This holds for all $T > r$, which, if $p(r,z;T,y)$ provides a sufficiently rich class of functions as we vary T, implies the result.

6. Suppose that $(X_t)_{t\geq 0}$ solves

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t,$$

where W is a Brownian motion. For $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ given deterministic functions, find the partial differential equation satisfied by the function

1. $F(t,x) \triangleq \mathbb{E}\left[\exp\left(-\int_t^T k(s,X_s)ds\right)\Phi(X_T)\middle| X_t = x\right]$,
2. $F(t,x) = \mathbb{E}\left[\Phi(X_T)\middle| X_t = x\right] + \int_t^T \mathbb{E}\left[k(X_s)\middle| X_t = x\right]ds$.

for $0 \leq t \leq T$.

[*You can assume that* k *and* Φ *are regular enough such the PDE is well-posed.*]

Solution: We start with (1) and evidently $F(T,x) = \Phi(x)$. By analogy with the proof of the Feynman-Kac representation, it is tempting to examine the dynamics of

$$Z_s = \exp\left(-\int_t^s k(u, X_u)du\right) F(s, X_s).$$

Notice that if this choice of $\{Z_s\}_{t \le s \le T}$ is a martingale we have that

$$Z_t = F(t,x) = \mathbb{E}\left[Z_T \middle| X_t = x\right].$$

Thus the partial differential equation satisfied by $F(t,x)$ is that for which $\{Z_t\}_{0 \le t \le T}$ is a martingale.

Our strategy now is to find the stochastic differential equation satisfied by $\{Z_s\}_{t \le s \le T}$. We proceed in two stages. Remember that t is now fixed and we vary s. First notice that

$$d\left(\exp\left(-\int_t^s k(u, X_u)du\right)\right) = -k(s, X_s)\exp\left(-\int_t^s k(u, X_u)du\right) ds$$

and by Itô's formula

$$\begin{aligned} dF(s,X_s) &= \frac{\partial F}{\partial s}(s,X_s)ds + \frac{\partial F}{\partial x}(s,X_s)dX_s + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(s,X_s)\sigma^2(s,X_s)ds \\ &= \left\{\frac{\partial F}{\partial s}(s,X_s) + \mu(s,X_s)\frac{\partial F}{\partial x}(s,X_s) + \frac{1}{2}\sigma^2(s,X_s)\frac{\partial^2 F}{\partial x^2}(s,X_s)\right\} ds \\ &\qquad + \sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_s. \end{aligned}$$

Hence

$$\begin{aligned} dZ_s = \exp\left(-\int_t^s k(u, X_u)du\right) \times \\ \left\{\left\{-k(s,X_s)F(s,X_s) + \frac{\partial F}{\partial s}(s,X_s) + \mu(s,X_s)\frac{\partial F}{\partial x}(s,X_s) + \frac{1}{2}\sigma^2(s,X_S)\frac{\partial^2 F}{\partial x^2}\right\} ds \right. \\ \left. + \sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_s\right\}. \end{aligned}$$

We can now read off the solution: $\{Z_s\}_{t \le s \le T}$ will be a martingale if F satisfies

$$\frac{\partial F}{\partial s}(s,x) + \mu(s,x)\frac{\partial F}{\partial x}(s,x) + \frac{1}{2}\sigma^2(s,x)\frac{\partial^2 F}{\partial x^2}(s,x) - k(s,x)F(s,x) = 0.$$

We now proceed to (2). Using the same reasoning, we apply Itô's formula to $F(s,X_s) + \int_t^s k(X_u)du$ and integrate with respect to s over $[t,T]$ to see that

$$\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} + k = 0,$$

and $F(T,x) = \Phi(x)$.

7. Let $X_1, X_2, \dots$ be independent and identically distributed random variables such that $\mathbb{P}(X_i = e) = \frac{1}{2d}$ for every $e = (e_1, \dots, e_d) \in \{-1, 0, 1\}^d$ with $\sum_{i=1}^d |e_i| = 1$. For $x \in \mathbb{Z}^d$ define $S^x = (S_n^x)_{n \geq 0}$ where for $n = 1, 2, \dots$

$$S_n^x = x + X_1 + \cdots + X_n$$

and $S_0^x = x$. Let $A \subset \mathbb{Z}^d$ be a finite set with boundary

$$\partial A = \{x \notin A : |x - y| = 1 \text{ for some } y \in A\}$$

and denote the first exit time as $\tau^x = \inf\{j \geq 0 : S_j^x \notin A\}$. The discrete Laplacian of a function $f : \mathbb{Z}^d \to \mathbb{R}$ is defined as

$$\Delta f(x) = \mathbb{E}[f(S_1^x) - f(S_0^x)].$$

We call a function f harmonic on A if $\Delta f(x) = 0$ for all $x \in A$.

1. Assume that $f : \mathbb{Z}^d \to \mathbb{R}$ is bounded and harmonic on A. Show that $M_n = f(S^x_{\min(n,\tau^x)})$ is a martingale with respect to the filtration generated by S^x.
2. Show that there exists a constant $c < \infty$ and a $\rho < 1$ such that for each $x \in A$ and $n \geq 0$

$$\mathbb{P}(\tau^x \geq n) \leq c\rho^n.$$

[*Hint: For* $R = \sup\{|x| : x \in A\}$ *and every* $x \in A$ *there is a path of length* $R + 1$ *starting in* x *and ending outside of* A*.*]

3. Let $F : \partial A \to \mathbb{R}$ and $g : A \to \mathbb{R}$. Assume that $f : A \cup \partial A \to \mathbb{R}$ satisfies

$$\Delta f(x) = -g(x) \text{ for } x \in A,$$
$$f(x) = F(x) \text{ for } x \in \partial A.$$

Show that $f(x) = \mathbb{E}[F(S^x_{\tau^x}) + \sum_{j=0}^{\tau^x - 1} g(S_j^x)]$.

Solution:

1. By the Markov property

$$E[f(S_{n+1})|F_n] = E^{S_n}[f(S_1)] = f(S_n) + \Delta f(S_n)$$

Let $B_n = \{\tau > n\}$. Then $M_{n+1} = M_n$ on B_n^c and

$$\begin{aligned} E[M_{n+1}|F_n] &= E[M_{n+1}1_{B_n}|F_n] + E[M_{n+1}1_{B_n^c}|F_n] && (5)\\ &= E[f(S_{n+1})1_{B_n}|F_n] + E[M_n 1_{B_n^c}|F_n] && (6)\\ &= 1_{B_n}E[f(S_{n+1})|F_n] + M_n 1_{B_n^c} && (7)\\ &= 1_{B_n}(f(S_n) + \Delta f(S_n)) + M_n 1_{B_n^c} && (8) \end{aligned}$$

But $\Delta f(S_n) = 0$ on B_n, therefore

$$E[M_{n+1}|F_n] = 1_{B_n} f(S_n) + 1_{B_n^c} M_n = M_n \tag{9}$$

2. Let $R = \sup\{|x| : x \in A\}$. Then for each $x \in A$ there is a path of length $R+1$ starting at x and ending outside of A, hence

$$P^x(\tau \leq R+1) \geq (2d)^{-(R+1)}.$$

By the Markov property

$$P^x(\tau > k(R+1)) = P^x(\tau > (k-1)(R+1)) P^x(\tau > k(R+1)|\tau > (k-1)(R+1)) \tag{10}$$

$$\leq P^x(\tau > (k-1)(R+1))(1-(2d)^{-(R+1)}). \tag{11}$$

and hence

$$P^x(\tau > k(R+1)) \leq \rho^{k(R+1)},$$

where $\rho = (1-(2d)^{-(R+1)})^{1/(R+1)}$. For an integer n write $n = k(R+1)+j$ where $j \in \{1, \ldots, R+1\}$. Then

$$P^x(\tau \geq n) \leq P^x(\tau > k(R+1)) \leq \rho^{k(R+1)} \leq \rho^{-(R+1)}\rho^n \tag{12}$$

3. First note that by above question, f is well-defined since

$$E^x[\sum_{j=0}^{\tau-1} |g(S_j)|] \leq \|g\|_\infty E^x[\tau] < \infty.$$

It is immediate to check that f as given satsifies the discrete PDE with boundary conditions. To check the uniqueness, assume f solves the discrete PDE and let M be the martingale

$$M_n = f(S_{n\wedge\tau}) - \sum_{j=0}^{(n-1)\wedge(\tau-1)} \Delta f(S_j) \tag{13}$$

$$= f(S_{n\wedge\tau} + \sum_{j=0}^{(n-1)\wedge(\tau-1)} g(S_j). \tag{14}$$

By the bounds of the previous question

$$E^x[|M_n|1_{\tau\geq n}] \leq (\|f\|_\infty + n\|g\|_\infty)P^x(\tau \geq n) \to 0.$$

Therefore the optimal sampling theorem applies and

$$f(x) = E^x[M_0] = E^x[M_\tau] = E^x[F(S_\tau) + \sum_{j=0}^{\tau-1} g(S_j)]$$

Section C

8. The *Vasicek model* models the interest rate $(r_t)_{t\geq 0}$ as solution of the stochastic differential equation

$$dr_t = (b - ar_t)dt + \sigma dW_t,$$

where W is standard Brownian motion. Find the Kolmogorov backward and forward differential equations satisfied by the probability density function of r_t. What is the distribution of r_t as $t \to \infty$?

Solution:

$$\frac{\partial p(t,T;x,y)}{\partial t} = -\frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial x^2} - (b - ax)\frac{\partial p}{\partial x}.$$

$$\frac{\partial p(t,T;x,y)}{\partial T} = \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial y^2} - \frac{\partial}{\partial y}\left((b - ay)p\right).$$

Consider $u_t = e^{at}r_t$.

$$du_t = be^{at}dt + \sigma e^{at}dW_t.$$

Integrating and substituting back gives

$$r_t = e^{-at}r_0 + e^{-at}\int_0^t be^{as}ds + \int_0^t \sigma e^{-a(t-s)}dW_s.$$

Thus r_t is normally distributed with mean $e^{-at}r_0 + \frac{b}{a}(1-e^{-at})$ and variance $\frac{\sigma^2}{2a}(1-e^{-2at})$. As $t \to \infty$, r_t tends to a normally distributed random variable with mean b/a and variance $\sigma^2/2a$.

9. The process usually known as *Geometric Brownian motion* solves the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Find the forward and backward Kolmogorov equations for geometric Brownian motion and show that the transition density for the process is the lognormal density given by

$$p(t,x;T,y) = \frac{1}{\sigma y\sqrt{2\pi(T-t)}}\exp\left(-\frac{\left(\log(y/x) - \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right).$$

Solution: Substituting in our formula for the forward equation we obtain

$$\frac{\partial p}{\partial T}(t,x;T,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(y^2 p(t,x;T,y)\right) - \mu\frac{\partial}{\partial y}\left(yp(t,x;T,y)\right),$$

and the backward equation is

$$\frac{\partial p}{\partial t}(t,x;T,y) = -\frac{1}{2}\sigma^2x^2\frac{\partial^2 p}{\partial x^2}(t,x;T,y) - \mu x\frac{\partial p}{\partial x}(t,x;T,y).$$

It is enough to check that the lognormal density solves one of the Kolmogorov equations.