Linear Theory

(1) $\quad Lu = A_0 \partial_t u + \sum_{j=1}^{n} A_j \partial_j u + Bu = F \quad$ in Q_T

$Q_T = (0,T) \times \Omega \qquad \Omega \subset \mathbb{R}^n \qquad t \in [0,T]$

$A_j = A_j(t,x) \qquad B = B(t,x)$

$A_j, B \in M_{N \times N}$

$u = u(t,x) \in \mathbb{R}^N, \qquad F = F(t,x) \in \mathbb{R}^N$

Def. (1) is symmetric hyperbolic system if

(i) $A_0, \dots, A_n$ symmetric

(ii) $A_0 > 0$

Def (1) is Friedrichs symmetrizable if

$\exists \quad S = S(t,x) \in M_{N \times N}$ (symmetrizer)

s.t.

$SA_0 \partial_t u + \sum_{j=1}^{n} SA_j \partial_j u + SB = SF$

is symm. hyp.

Conservation of energy $\qquad \Omega = \mathbb{R}^n$, $\quad \Omega = \mathbb{T}^n$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (A_0 u, u)_{\mathbb{R}^N}\, dx = \int_\Omega (F, u)\, dx + \frac{1}{2}\int_\Omega \left((\mathrm{Div}\,\vec{A} - 2B)u, u\right)$$

$$\mathrm{Div}\,\vec{A} = \partial_t A_0 + \sum_j \partial_j A_j$$

$\partial\Omega \neq \emptyset$

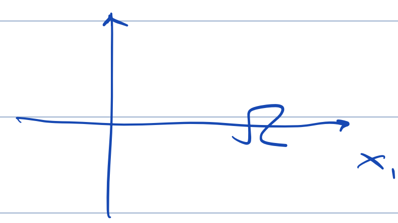
b.c. $\qquad Mu = G \qquad$ on $\quad \Sigma_T = (0,T)\times\partial\Omega$

$M = M(t,x) \in \mathbb{M}_{d\times N} \qquad$ rank $M = d \quad$ maximal rank

$d = \#$ b.c.

$\Omega = \{x_1 > 0\}$

$\partial\Omega = \{x_1 = 0\}$



$F = F(t, x_1) \qquad G(t) \qquad u_{|t=0} = u_0(x_1)$

$u = u(t, x_1)$

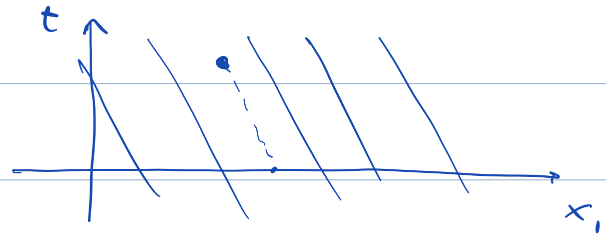
$$\begin{cases} A_0 \partial_t u + A_1 \partial_1 u = F & x_1 > 0 \\ Mu(0,t) = G(t) & x_1 = 0 \\ u(x_1, 0) = u_0(x_1) & \end{cases}$$

$A_0 = \mathbb{I}$ $\quad A_1, M$ constant

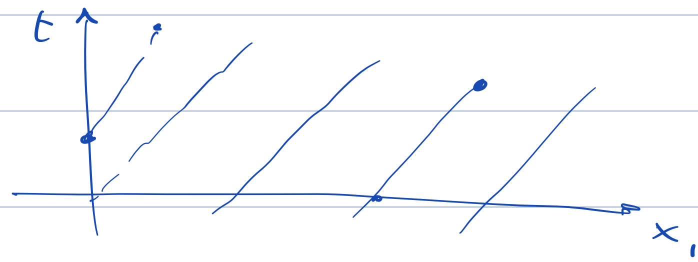
$$A_1 = \mathrm{diag}(a_1, \dots, a_N) \qquad a_1 \geq a_2 \geq \dots \geq a_N$$

$$(\partial_t + a_j \partial_1) u = F_j$$

$a_j < 0$



$a_j > 0$



$$\dots \geq a_p > 0 \geq a_{p+1} \geq \dots$$

"p" boundary conditions

$$d = \mathrm{rank}(M) = p$$

d = # positive eigenvalues of A_1

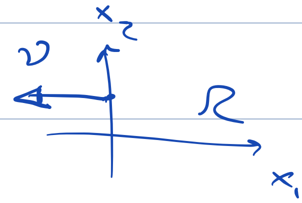
$$A_\nu = \sum_{j=1}^n A_j \nu_j \qquad \nu = (\nu_1, \dots, \nu_n)$$

unit outward normal to $\partial\Omega$

Boundary matrix

$d = \#$ negative eigenvalues of A_ν

$\Omega = \{ x_1 > 0 \}$



$\nu = (-1, 0, \dots, 0)$

$$A_\nu = - A_1$$

I.B.V.P

$$(3) \quad \begin{cases} Lu = F & \text{in } Q_T \\ Mu = G & \text{on } \Sigma_T \\ u|_{t=0} = u_0 & \text{in } \Omega \end{cases}$$

<u>Def.</u> $\partial\Omega$ is <u>non-characteristic</u> for L if A_ν is invertible at $\partial\Omega$

($\det A_\nu \neq 0$)

<u>Def.</u> Assume the bdry is non-characteristic

The b.c. is <u>strictly dissipative</u> if

$\exists \ \delta > 0, \ c > 0$ s.t.

$$(A_\nu u, u) \geq \delta |u|^2 - c |Mu|^2 \qquad \forall u$$

$\forall (t,x) \in \Sigma_T$

Equivalent to $\quad A_\nu > 0 \quad$ on $\ker M$:

(i) $\quad u \neq 0, \quad Mu = 0 \quad \Rightarrow \quad (A_\nu u, u) > 0$

(ii) $\quad \ker M \quad$ maximal for (i)

(iii) $\quad M$ is onto $\quad$ ($\Leftrightarrow$ rank(M) maximal)

$$\underset{\wedge}{\delta |u|^2 - c|Mu|^2}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (A_0 u, u)_{\mathbb{R}^N}\, dx \quad + \frac{1}{2}\int_{\partial\Omega} (A_\nu u, u)\, dS_x$$

$$= \int_\Omega (F, u)\, dx + \frac{1}{2}\int_\Omega ((\mathrm{Div}\vec{A} - 2B)u, u)$$

$$\frac{d}{dt}\int_\Omega (A_0 u, u)\, dx \quad + \quad \delta\int_{\partial\Omega} |u|^2\, dS_x \quad \leq \quad c\int_{\partial\Omega} |G|^2\, dS_x$$

$$+ \int_\Omega (F, u)\, dx + \frac{1}{2}\int_\Omega ((\mathrm{Div}\vec{A} - 2B)u, u)$$

$\Rightarrow \quad u(t) \in L^2(\Omega) \qquad\qquad u \in L^\infty(0,T; L^2(\Omega))$

$\qquad u(t)_{|\partial\Omega} \in L^2(\Sigma_T)$

regular solution $\quad$ ($u_0 \in H^m(\Omega)$ + comp. cond's)

$\qquad u(t) \in H^m(\Omega) \qquad \forall t$

$\qquad u(t)_{|\Sigma_T} \in H^m(\Sigma_T)$

$$\Rightarrow \quad u \in \bigcap_{k=0}^{m} C^k([0,T]; H^{m-k}(\Omega))$$

$u(t) \in H^m(\Omega)$ "full regularity"

(i) tangential differentiation along the bdry

tang. derivatives satisfy similar b.c. as u

(ii) $\underset{\neq 0}{A_\nu}\, \partial_\nu u = F - (A_0 \partial_t u + A_{tan} \partial_{tan} u + B')$

Def. The bdry $\partial\Omega$ is <u>characteristic</u> for L is A_ν is singular at $\partial\Omega$

- $A_\nu \equiv 0$ at $\partial\Omega$ $\Rightarrow$ L tang. operator

- A_ν constant rank in a neighborhood of $\partial\Omega$

  <u>uniformly characteristic</u> Majda-Osher '75

- A_ν <u>constant rank</u> at $\partial\Omega$

  characteristic bdry of <u>constant multiplicity</u>

  $\dim \ker A_\nu = \text{const}$

  Rauch '85

- The rank of A_ν is not constant on $\partial\Omega$

nonuniformly charact. Rauch '85

constant multiplicity ...

strictly dissipative b.c.

$$(A_\nu u, u) \geq \delta |Pu|^2 - c|Mu|^2$$

P orthogonal projection onto $(\ker A_\nu)^\perp$

$$A_\nu = \begin{bmatrix} A_\nu^{II} & A_\nu^{I\,II} \\ A_\nu^{II\,I} & A_\nu^{II\,II} \end{bmatrix}$$

A_ν^{II} invertible

$A_\nu^{I\,II} = 0$ at $\partial\Omega$

$A_\nu^{II\,I} = 0$, $A_\nu^{II\,II} = 0$

$$u = \begin{pmatrix} u^{I} \\ u^{II} \end{pmatrix} \qquad Pu = \begin{pmatrix} u^{I} \\ 0 \end{pmatrix}$$

Def b.c. are maximally non-negative (dissipative) of A_ν if

(i) $(A_\nu u, u) \geq 0 \quad \forall (t,x) \in \Sigma_T, \ \forall u \in \ker M(t,x)$

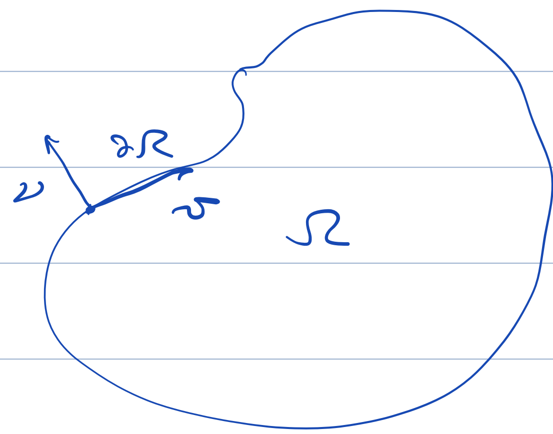
(ii) $\ker M$ is not properly contained in any other subspace with (i)

$$\int_{\partial\Omega} (A_\nu u, u)\, dS_x \geq 0$$

Non uniformly b.c. rank M not const at $\partial\Omega$

MHD with a perfectly conducting wall
Yanagisawa - Matsumura '91

$v \cdot \nu = 0$ at $\partial\Omega$



$\sigma = \infty \Longrightarrow E \times \nu = 0$

$J = \sigma (E + v \times H)$

$\Longrightarrow E = - v \times H$

$(v \times H) \times \nu = 0$

$\underbrace{(v \cdot \nu)}_{=0} H - (H \cdot \nu) v = 0$

$$\begin{cases} v\cdot\nu = 0 \\ (H\cdot\nu)\,v = 0 \end{cases}$$

$$H(t)\cdot\nu = H_0\cdot\nu$$

$\Longleftrightarrow$ $v\cdot\nu = 0$ on $\Gamma_0 = \{x\in\partial\Omega : H_0(x)\cdot\nu = 0\}$

$v = 0$ on $\Gamma_1 = \{x\in\partial\Omega : H_0(x)\cdot\nu \neq 0\}$

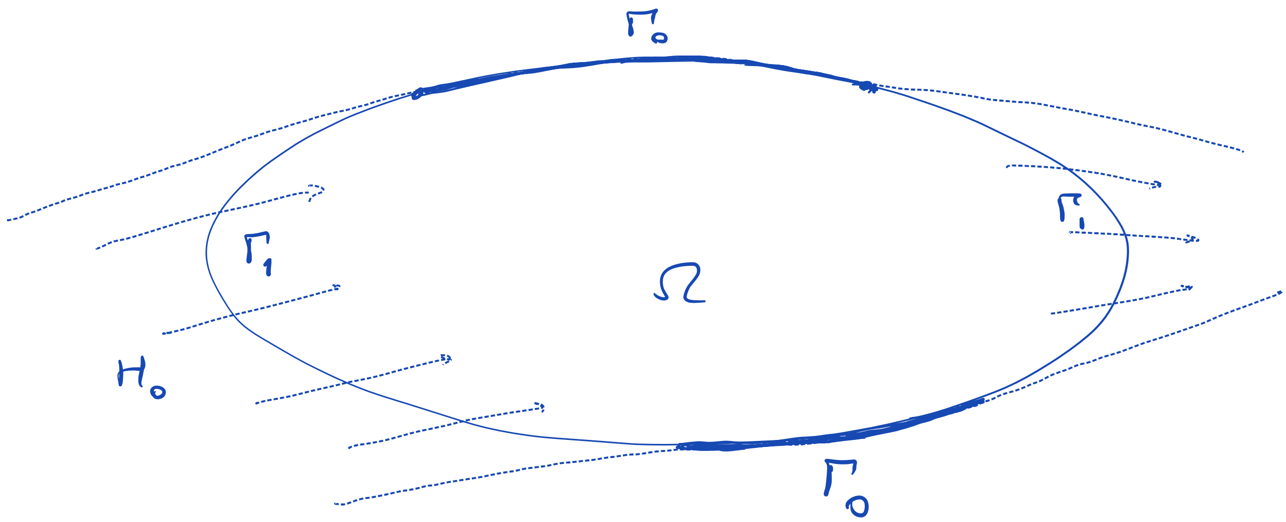
$\partial\Omega = \Gamma_1$ T. Shirota, Yanagisawa-Matsumura '81

$\partial\Omega = \Gamma_0$ Y-M

MHD   Perfectly conducting wall

$v \cdot \nu = 0$ on $\Gamma_0 = \{ x \in \partial\Omega : H_0(x) \cdot \nu = 0 \}$

$v = 0$ on $\Gamma_1 = \{ x \in \partial\Omega : H_0(x) \cdot \nu \neq 0 \}$



$\partial\Omega = \Gamma_1$ T. Shizuta, T. Yanagisawa - A. Matsumura '91

rank $A_\nu = 6$

$\partial\Omega = \Gamma_0$ T. Yanagisawa - A. Matsumura '91

rank $A_\nu = 2$

If $\dim \ker A_\nu$ is not constant on $\partial\Omega$,
then weak solutions are not necessarily strong
(ill-posedness)

Phillips-Sarason '66

Moyer '68

Osher '73

Rauch '94

good strategy:

add boundary conditions from one part of the boundary to the other part

$\Longrightarrow$ weak = strong (in $L^2(\Omega)$)

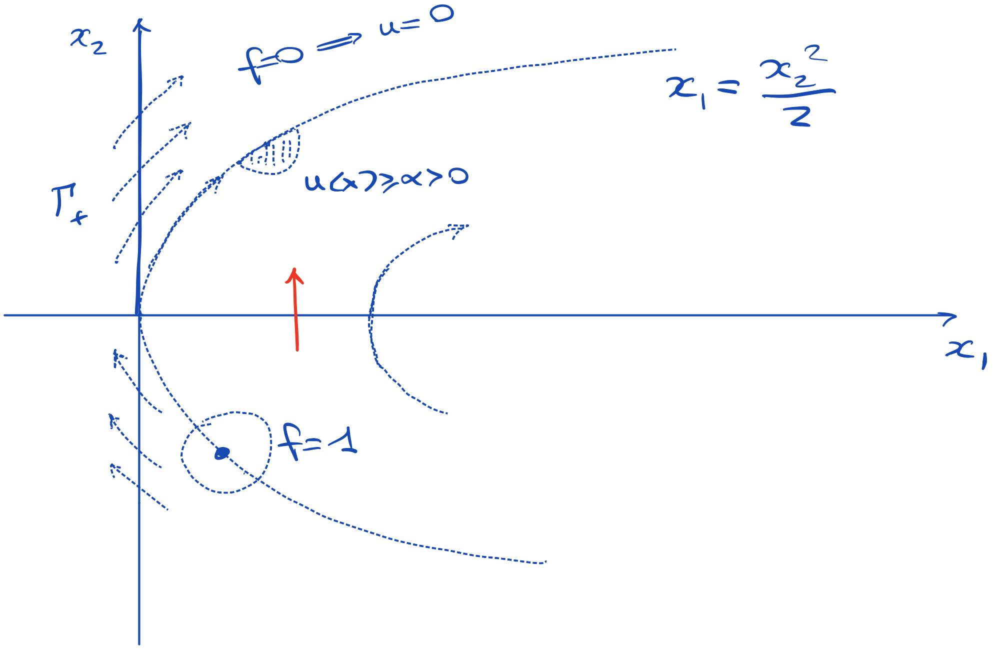
Next problem:

regularity of L^2- strong solutions

$\Omega = \mathbb{R}^2_+ = \{x_1 > 0\}$, $\Gamma_+ = \{x_1 = 0, x_2 > 0\}$, $\lambda > 0$ large

$a(x) \in C^\infty_{(0)}(\Omega)$, $a(x) \geq 0$, $a(x) = 1$ for $|x| \leq R$

$$\begin{cases} (\lambda + x_2 a(x) \partial_{x_1} + \partial_{x_2}) u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_+ \end{cases}$$

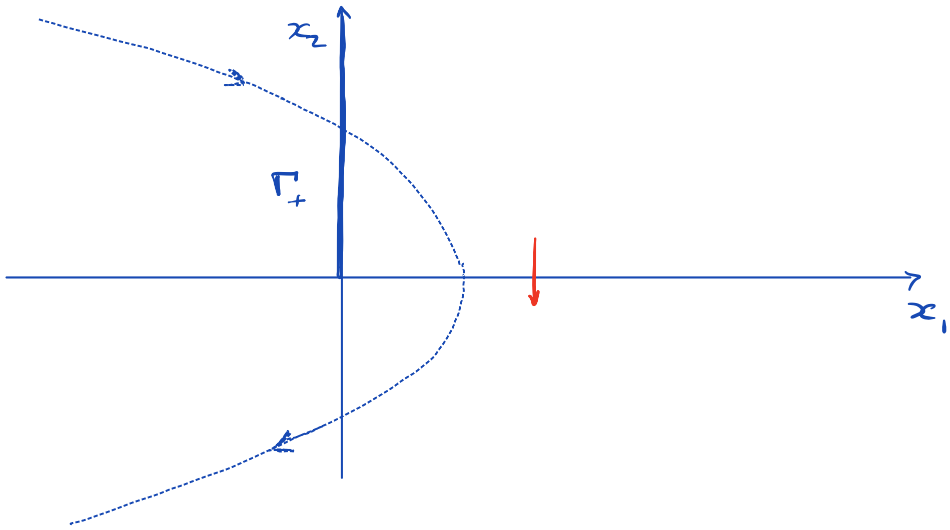


$f \in C^\infty_{(0)}(\Omega)$, $0 \leq f \leq 1$

$u \notin H^1(\Omega)$

Need a transversal sign condition!

$$\begin{cases} (x_2\, a(x)\partial_{x_1} - \partial_{x_2})u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_+ \end{cases}$$



Need vanishing conditions on f !

$f \in C^\infty_{(0)}(\Omega)$, $f = 1$ in a neighborhood of $(0,0)$

$$\Longrightarrow \quad \partial^2_{11} u = -(2x_1 + x_2^2)^{-3/2} \quad \text{near } (0,0)$$

$$\notin L^2(\Omega)$$

$$\Longrightarrow \quad u \notin H^2(\Omega)$$

Nishitani – Takayama '96 , 2000

Secchi '98 , 2000