Characteristic IBVP's and Magneto-Hydrodynamics

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Plan

1. Introduction, definitions
2. Characteristic IBVP for hyperbolic systems
   - Examples: Euler equations, MHD
   - Anisotropic Sobolev spaces and MHD
3. Kreiss-Lopatinskii condition
   - Analysis of Majda's example

Kreiss-Lopatinskii condition

Consider the BVP

$$\begin{cases} Lu = F\,, & \text{in}\,\{x_1 > 0\}\,, \\ Mu = G\,, & \text{on}\,\{x_1 = 0\}\,. \end{cases} \tag{9}$$

- $L := \partial_t + \sum_{j=1}^{n} A_j \partial_{x_j}$, hyperbolic operator (with eigenvalues of constant multiplicity);
- $A_j \in \mathbf{M}_{N\times N}$, $j = 1, \ldots, n$, and $\det A_1 \neq 0$ (i.e. non characteristic boundary);
- $M \in \mathbf{M}_{d\times N}$, rank$(M) = d = \#\{\text{positive eigenvalues of } A_1\}$.

- Let $u = u(x_1, x', t)$ ($x' = (x_2, \dots, x_n)$) be a solution to (9) for $F = 0$ and $G = 0$.
- Let $\widehat{u} = \widehat{u}(x_1, \eta, \tau)$ be Fourier-Laplace transform of u w.r.t. x' and t respectively (η and τ dual variables of x' and t respectively).
- $\widehat{u}$ solves the ODE problem

$$\begin{cases} \frac{d\widehat{u}}{dx_1} = \mathcal{A}(\eta,\tau)\widehat{u}\,, \quad x_1 > 0\,, \\ M\widehat{u}(0) = 0\,, \end{cases} \tag{10}$$

where $\mathcal{A}(\eta,\tau) := -(A_1)^{-1}\left(\tau I_n + i\sum_{j=2}^{n} A_j\eta_j\right)$.

Let $\mathcal{E}^-(\eta,\tau)$ be the stable subspace of (10).

- Kreiss-Lopatinskii condition (KL):

$$ker M \cap \mathcal{E}^-(\eta,\tau) = \{0\}, \quad \forall(\eta,\tau) \in \mathbb{R}^{n-1} \times \mathbb{C},\ \Re\tau > 0.$$

$$\Updownarrow$$

$$\forall(\eta,\tau) \in \mathbb{R}^{n-1} \times \mathbb{C},\ \Re\tau > 0,\ \exists C = C(\eta,\tau) > 0 : \\ |A_1 V| \le C|MV| \quad \forall V \in \mathcal{E}^-(\eta,\tau).$$

- Uniform Kreiss-Lopatinskii condition (UKL):

$$\exists C > 0 : \ \forall(\eta,\tau) \in \mathbb{R}^{n-1} \times \mathbb{C},\ \Re\tau > 0 : \\ |A_1 V| \le C|MV| \quad \forall V \in \mathcal{E}^-(\eta,\tau).$$

Lopatinskii determinant

• For all $(\eta,\tau)\in\mathbb{R}^{n-1}\times\mathbb{C}$, $\Re\tau>0$, let $\{X_1(\eta,\tau),\dots,X_d(\eta,\tau)\}$ be an orthonormal basis of $\mathcal{E}^-(\eta,\tau)$ $(\dim\mathcal{E}^-(\eta,\tau)=\operatorname{rank}M=d)$.
• Constant multiplicity of the eigenvalues $\Rightarrow X_j(\eta,\tau)$, $j=1,\dots,d$, and $\mathcal{E}^-(\eta,\tau)$ can be extended to all $(\eta,\tau)\neq(0,0)$ with $\Re\tau=0$.

$$\Delta(\eta,\tau):=\det\left[M\left(X_1(\eta,\tau),\dots,X_d(\eta,\tau)\right)\right]$$
$$\forall(\eta,\tau)\in\mathbb{R}^{n-1}\times\mathbb{C},\ \Re\tau\geq0.$$

$$(KL)\quad\Leftrightarrow\quad\Delta(\eta,\tau)\neq0\,,\quad\forall\Re\tau>0,\forall\eta\in\mathbb{R}^{n-1}\,.$$

$$(UKL)\quad\Leftrightarrow\quad\Delta(\eta,\tau)\neq0\,,\quad\forall\underline{\Re\tau\geq0},\forall\eta\in\mathbb{R}^{n-1}\,.$$

Kreiss-Lopatinskii condition and well posedness

1. $\boxed{\det A_1 \neq 0}$ (i.e. non characteristic boundary)
   - NOT (KL) $\Rightarrow$ (9) is ill posed in Hadamard's sense;
   - (UKL) $\Leftrightarrow$ L^2-strong well posedness of (9);
   - (KL) but NOT (UKL) $\Rightarrow$ Weak well posedness of (9) (energy estimate with loss of regularity ?).
2. $\boxed{\det A_1 = 0}$ (i.e. characteristic boundary)
   - NOT (KL) $\Rightarrow$ (9) is ill posed in Hadamard's sense;
   - (UKL) + structural assumptions on L $\Rightarrow$ L^2-strong well posedness of (9).

Structural assumptions

- [Majda & Osher, 1975]:
  1. L symmetric hyperbolic, with variable coefficients +
  2. Uniformly characteristic boundary +
  3. (UKL) +
  4. Several structural assumptions on L and M, among which that:

$$A(\eta) := \sum_{j=2}^{n} A_j \eta_j = \begin{pmatrix} a_1(\eta) & a_{2,1}(\eta)^T \\ a_{2,1}(\eta) & a_2(\eta) \end{pmatrix}$$

where $a_1(\eta)$ has only simple eigenvalues for $|\eta| = 1$.

Satisfied by: strictly hyperbolic systems, MHD, Maxwell's equations, linearized shallow water equations.

NOT satisfied by: 3D isotropic elasticity ($a_1(\eta) = 0_3$).

- [Benzoni-Gavage & Serre, 2003]:
  1. L symmetric hyperbolic, with constant coefficients, M constant $+$
  2. (Uniformly) characteristic boundary, $ker\, A_\nu \subset ker\, M$ $+$
  3. (UKL) $+$
  4. $$A(\eta) = \begin{pmatrix} 0 & a_{2,1}(\eta)^T \\ a_{2,1}(\eta) & a_2(\eta) \end{pmatrix}$$

     with $a_2(\eta) = 0$.

  Satisfied by: Maxwell's equations, linearized acoustics.
  NOT satisfied by: isotropic elasticity ($a_2(\eta) \neq 0$).
- [Morando & Serre, 2005]: $2D, 3D$ linear isotropic elasticity.

Majda's example

Initial-boundary value problem for the scalar wave equation:

$$\begin{cases} U_{tt} - U_{xx} - U_{yy} = 0 & \text{for } t > 0,\ x \in \mathbb{R},\ y > 0, \\ \Gamma U_t + U_y = 0 & \text{for } y = 0\,, \\ i.c. & \text{for } t = 0\,, \end{cases} \qquad (1)$$

where $\Gamma \in \mathbb{R}$ is a parameter.

Problem (1) was first introduced by A. Majda[1].

[1] Compressible fluid flow and systems of conservation laws in several space variables, vol. 53 Appl. Math. Sciences, Springer-Verlag, NY 1984.

Energy method

Total energy

$$E(t) := \frac{1}{2} \int_{\mathbb{R}} \int_0^{\infty} \left(U_t^2 + U_x^2 + U_y^2\right)\, dxdy$$

Multiply $(1)_1$ by U_t and integrate:

$$\frac{d}{dt} E(t) = -\int_{y=0} U_t U_y \, dx = \Gamma \int_{y=0} U_t^2 \, dx$$

Then

- $\Gamma < 0$: the boundary condition removes energy (stabilizing effect)
- $\Gamma > 0$: the boundary condition adds energy (instability ???)

Boundary value problem

Reduce (1) to the boundary value problem for the scalar wave equation:

$$\begin{cases} U_{tt} - U_{xx} - U_{yy} = 0 & \text{for } t \in \mathbb{R},\ x \in \mathbb{R},\ y > 0, \\ \Gamma U_t + U_y = g & \text{for } y = 0\,. \end{cases} \tag{2}$$

Introduce the new unknowns:

$$v := U_t\,, \quad w := -U_x, \quad z := -U_y.$$

In terms of (v, w, z) problem (2) gives the **Euler-type system**

$$\begin{cases} v_t + w_x + z_y = 0\,, \\ w_t + v_x = 0\,, \\ z_t + v_y = 0 & y > 0\,, \\ \Gamma v - z = g & y = 0\,. \end{cases} \tag{3}$$

In fact, we can write the system (3)

$$\begin{cases} v_t + w_x + z_y = 0\,, \\ w_t + v_x = 0\,, \\ z_t + v_y = 0 & y > 0\,, \\ \Gamma v - z = g & y = 0\,, \end{cases}$$

in vector form as the “acoustic system”

$$\begin{cases} v_t + \operatorname{div}_{x,y} \cdot \begin{pmatrix} w \\ z \end{pmatrix} = 0\,, \\ \partial_t \begin{pmatrix} w \\ z \end{pmatrix} + \nabla v = 0\,, & y > 0\,, \\ \Gamma v - z = g & y = 0\,. \end{cases}$$

Second formulation of the problem

Let us introduce the new unknown $u = (u_1, u_2, u_3)^\mathsf{T}$ defined by

$$u_1 = w, \quad u_2 = \frac{1}{2}(z - v), \quad u_3 = \frac{1}{2}(z + v),$$

that is

$$u_1 = -U_x, \quad u_2 = -\frac{1}{2}(U_t + U_y), \quad u_3 = \frac{1}{2}(U_t - U_y).$$

In terms of u the Euler-type problem (3) reads

$$\begin{aligned} &\begin{pmatrix} \partial_t & -\partial_x & \partial_x \\ -\partial_x & 2(\partial_t - \partial_y) & 0 \\ \partial_x & 0 & 2(\partial_t + \partial_y) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 && \text{if } y > 0\,, \\ &-(\Gamma + 1)u_2 + (\Gamma - 1)u_3 = g && \text{if } y = 0\,. \end{aligned} \tag{4}$$

Denote by $\widehat{u}$ the Laplace-Fourier transforms of u in (t,x), with dual variables $\tau = \gamma + i\delta$ and η, for $\gamma \geq 1$ and $\delta, \eta \in \mathbb{R}$. We obtain from (4)

$$\begin{pmatrix} \tau & -i\eta & i\eta \\ i\eta & 2(\frac{\mathrm{d}}{\mathrm{d}y} - \tau) & 0 \\ i\eta & 0 & 2(\frac{\mathrm{d}}{\mathrm{d}y} + \tau) \end{pmatrix} \widehat{u} = 0 \qquad \text{if } y > 0\,, \tag{5a}$$

$$\beta \widehat{u^{\mathrm{nc}}} = \widehat{g} \qquad \text{if } y = 0\,, \tag{5b}$$

where

$$\beta = (-(\Gamma+1), \Gamma - 1), \qquad u^{\mathrm{nc}} = (u_2, u_3)^{\mathsf{T}}.$$

From the first (algebric) equation of (5a) we express $\widehat{u}_1$ in terms of $\widehat{u}_2, \widehat{u}_3$ and plug the resulting expression into the other two equations of (5a).

We obtain a system of O.D.E.s:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}y}\widehat{u^{\mathrm{nc}}} = \mathcal{A}(\tau,\eta)\widehat{u^{\mathrm{nc}}} & \text{if } y > 0\,, \\ \beta\widehat{u^{\mathrm{nc}}} = \widehat{g} & \text{if } y = 0\,. \end{cases} \tag{6}$$

Here

$$\mathcal{A}(\tau,\eta) := \begin{pmatrix} \mu & -m \\ m & -\mu \end{pmatrix}, \qquad \mu := \tau + m, \qquad m := \frac{\eta^2}{2\tau}\,.$$

- $\mathcal{A}(\tau,\eta)$ is (positively) **homogeneous** of degree 1 in (τ,η). To take this homogeneity into account, we define the hemisphere:

$$\Xi_1 := \left\{(\tau,\eta) \in \mathbb{C}\times\mathbb{R} \,:\, \mathrm{Re}\,\tau \geq 0,\ |\tau|^2 + \eta^2 = 1\right\}.$$

- The **poles** of symbol $\mathcal{A}(\tau,\eta)$ on Ξ_1 are the points $(\tau,\eta) = (0,\pm 1) \in \Xi_1$ (where the coefficient of $\widehat{u}_1$ in the first equation of (5a) vanishes).
- We set

$$\Xi := (0,\infty)\cdot\Xi_1\,.$$

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- We set

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Lopatinskiĭ condition

Stability / instability of (6) is detected by the Lopatinskiĭ condition.

$$\omega := -\sqrt{\tau^2+\eta^2} = \begin{cases} \text{eigenvalue of } \mathcal{A}(\tau,\eta) \text{ with } \textbf{negative} & \\ \text{real part,} & \operatorname{Re}\tau > 0, \\ \text{continuous extension,} & \operatorname{Re}\tau = 0. \end{cases}$$

$$E(\tau,\eta) := \left(\frac{\eta^2}{2}, \tau(\mu-\omega)\right)^{\mathsf{T}} \text{ eigenvector of } \mathcal{A}(\tau,\eta) \text{ corresponding to } \omega$$

Definition

- The Lopatinskiĭ “determinant” associated to (6) is defined by

$$\Delta(\tau,\eta) := \det\left[\beta\, E(\tau,\eta)\right] = (\tau-\omega)(\Gamma\tau+\omega). \tag{7}$$

- We say that the Lopatinskiĭ condition holds if
  $\Delta(\tau,\eta) \neq 0$ for all $(\tau,\eta) \in \Xi_1$ with $\operatorname{Re}\tau > 0$;
- We say that the **uniform** Lopatinskiĭ condition holds if
  $\Delta(\tau,\eta) \neq 0$ for all $(\tau,\eta) \in \Xi_1$.

Definition

- If the Lopatinskiĭ condition is not satisfied the problem is said **violently unstable** (Hadamard ill-posedness).
- If the uniform Lopatinskiĭ condition holds then the problem is said **uniformly stable**.
- If the Lopatinskiĭ condition holds but not uniformly the problem is said **weakly stable**.

Lemma [Lopatinskiĭ condition for (6)]

(1) $\Gamma < 0$. Then $\Delta(\tau, \eta) \neq 0$ for every $(\tau, \eta) \in \Xi_1$. Problem (6) is **uniformly stable**.

(2) $0 \leq \Gamma < 1$. Let us define $\Lambda := (1 - \Gamma^2)^{-1/2}$. Then, for any $(\tau, \eta) \in \Xi_1$,

$$\Delta(\tau, \eta) = 0 \quad \text{if and only if} \quad \tau = \pm i \Lambda \eta \, .$$

Problem (6) is **weakly stable**.

(3) $\Gamma \geq 1$. Problem (6) is **violently unstable**.

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$$\Delta(\tau, \eta) = 0 \quad \text{if and only if} \quad \tau = \pm i \Lambda \eta \,.$$

Problem (6) is **weakly stable**.

(3) $\Gamma \geq 1$. Problem (6) is **violently unstable**.

The uniformly stable case $\Gamma < 0$

For $\tau = \gamma + i\delta$, where $\gamma \geq 1$ and $\delta, \eta \in \mathbb{R}$, set

$$\lambda(\tau, \eta) := (|\tau|^2 + \eta^2)^{\frac{1}{2}} = (\gamma^2 + \delta^2 + \eta^2)^{\frac{1}{2}}.$$

Introduce the **weighted Sobolev space**

$$H^s_\gamma(\mathbb{R}^2) := \left\{ u \in \mathcal{D}'(\mathbb{R}^2) \, : \, e^{-\gamma t} u \in H^s(\mathbb{R}^2) \right\},$$

$$\|u\|_{H^s_\gamma(\mathbb{R}^2)} := \frac{1}{2\pi} \|\lambda^s \widehat{e^{-\gamma t} u}\|_{L^2(\mathbb{R}^2)}, \qquad L^2_\gamma(\mathbb{R}^2) = H^0_\gamma(\mathbb{R}^2).$$

Theorem

Assume $\Gamma < 0$. For all $\gamma \geq 1$, if $u \in H^1(\mathbb{R}^3_+)$ is a solution to (4) the following estimate holds:

$$\gamma \|u\|^2_{L^2(\mathbb{R}^+; L^2_\gamma(\mathbb{R}^2))} + \|u^{\mathrm{nc}}|_{x_2=0}\|^2_{L^2_\gamma(\mathbb{R}^2)} \lesssim \|g\|^2_{L^2_\gamma(\mathbb{R}^2)}.$$

$\Longrightarrow$ **No loss of regularity** from the boundary datum.

PROOF

Because of the direct estimate

$$\gamma \|u\|^2_{L^2(\mathbb{R}^+;L^2_\gamma(\mathbb{R}^2))} \lesssim \|u^{\mathrm{nc}}|_{x_2=0}\|^2_{L^2_\gamma(\mathbb{R}^2)},$$

it's enough to show:

$$\|u^{\mathrm{nc}}|_{x_2=0}\|_{L^2_\gamma(\mathbb{R}^2)} \lesssim \|g\|_{L^2_\gamma(\mathbb{R}^2)}. \qquad (8)$$

Lemma

For all $(\tau_0, \eta_0) \in \Xi_1$, there exist a neighborhood $\mathscr{V}$ of (τ_0, η_0) in Ξ_1 and a continuous invertible matrix $T(\tau, \eta)$ defined on $\mathscr{V}$ such that

$$\forall (\tau, \eta) \in \mathscr{V} \setminus \underbrace{\{\tau = 0\}}_{\textbf{pole of } \mathcal{A}}, \quad T^{-1}\mathcal{A}T(\tau, \eta) = \begin{pmatrix} \omega & z \\ 0 & -\omega \end{pmatrix}.$$

The first column of $T(\tau, \eta)$ is $E(\tau, \eta)$.

Since Ξ_1 is compact, there exists a finite covering $\{\mathscr{V}_1, \dots, \mathscr{V}_J\}$ of Ξ_1 by such neighborhoods with corresponding matrices $\{T_1, \dots, T_J\}$, and a smooth **partition of unity** $\{\chi_j(\tau, \eta)\}_{j=1}^J \in C_c^\infty(\mathscr{V}_j)$ such that $\sum_{j=1}^J \chi_j^2 = 1$ on Ξ_1.

Define $\Pi_j := \{(\tau,\eta) \in \Xi : \exists\, s > 0,\ s \cdot (\tau,\eta) \in \mathscr{V}_j\}$ and

$$\mathsf{W}(\tau,\eta,y) := \chi_j T_j(\tau,\eta)^{-1}\widehat{u^{\mathrm{nc}}}(\tau,\eta,y), \quad \forall\,(\tau,\eta) \in \Pi_j.$$

Assume that $(\tau,\eta) \in \Pi_j$ and $\operatorname{Re}\tau > 0$. Then $\dfrac{\mathrm{d}\mathsf{W}}{\mathrm{d}y} = T_j^{-1}\mathcal{A}T_j\mathsf{W}$. Hence

$$\frac{\mathrm{d}\mathsf{W}_2}{\mathrm{d}y} = -\omega\mathsf{W}_2, \quad \Longrightarrow \mathsf{W}_2 = 0 \ \ (\operatorname{Re}\omega < \mathbf{0}).$$

Using the boundary equation (5b) ($\beta\widehat{u^{\mathrm{nc}}} = \widehat{g}$), one has

$$\chi_j\widehat{g} = \beta T_j(\tau,\eta)\mathsf{W}(\tau,\eta,0) = \underbrace{\beta E(\tau,\eta)}_{\Delta(\tau,\eta)}\mathsf{W}_1(\tau,\eta,0). \tag{9}$$

Because ($\Gamma < 0$: **uniform stability**)

$$\Delta(\tau,\eta) \neq 0 \qquad \forall(\tau,\eta) \in \Xi_1,$$

$$\exists\, C_1, C_2 > 0 : \quad C_1 \leq \Delta(\tau,\eta) \leq C_2 \qquad \forall(\tau,\eta) \in \Xi_1.$$

Extend $\Delta(\tau,\eta)$ as a homogeneous function of degree 0; then

$$C_1 \leq \Delta(\tau,\eta) \leq C_2 \qquad \forall(\tau,\eta) \in \Xi.$$

From (9)

$$|\mathsf{W}_1(\tau,\eta,0)| \lesssim \left|\chi_j \widehat{g}(\tau,\eta)\right|.$$

Therefore, for all $(\tau,\eta) \in \Pi_j$ with $\gamma = \operatorname{Re}\tau > 0$,

$$\left|\chi_j \widehat{u^{\mathrm{nc}}}(\tau,\eta,0)\right| \lesssim \left|\chi_j \widehat{g}(\tau,\eta)\right|.$$

Applying Plancherel's theorem yields

$$\|u^{\mathrm{nc}}|_{x_2=0}\|_{L^2_\gamma(\mathbb{R}^2)} \lesssim \|g\|_{L^2_\gamma(\mathbb{R}^2)},$$

that is (8). □

The uniformly stable case $\Gamma < 0$ (ibvp)

More in general, for the problem

$$\begin{cases} U_{tt} - U_{xx} - U_{yy} = F & \text{for } t \in \mathbb{R},\ x \in \mathbb{R},\ y > 0, \\ \Gamma U_t + U_y = 0 & \text{for } y = 0\,, \\ U = 0 & \text{for } t < 0\,, \end{cases} \tag{10}$$

where F is a given source term such that $F = 0$ for $t < 0$, one can obtain

Theorem

Assume $\Gamma < 0$. For all $m \geq 0$ and for $\gamma \geq 1$, if $u \in H^{m+1}_\gamma(\mathbb{R}^3_+)$ is a solution to (10) the following estimate holds:

$$\gamma \|u\|^2_{H^m_\gamma(\mathbb{R}^3_+)} + \|u^{\mathrm{nc}}|_{x_2=0}\|^2_{H^m_\gamma(\mathbb{R}^2)} \lesssim \|F\|^2_{H^m_\gamma(\mathbb{R}^3_+)}.$$

$\Longrightarrow$ **No loss of regularity** from the source term.

The weakly stable case $0 < \Gamma < 1$

Theorem

Assume $0 < \Gamma < 1$. For all $\gamma \geq 1$, if $u \in H^2(\mathbb{R}^3_+)$ is a solution of (4) the following estimate holds:

$$\gamma \|u\|^2_{L^2(\mathbb{R}+;L^2_\gamma(\mathbb{R}^2))} + \|u^{\mathrm{nc}}|_{x_2=0}\|^2_{L^2_\gamma(\mathbb{R}^2)} \lesssim \|g\|^2_{H^1_\gamma(\mathbb{R}^2)}.$$

$\Longrightarrow$ **Loss of regularity** from the boundary datum.

For the proof it's enough to show the estimate:

$$\|u^{\mathrm{nc}}|_{x_2=0}\|_{L^2_\gamma(\mathbb{R}^2)} \lesssim \|g\|_{H^1_\gamma(\mathbb{R}^2)}. \tag{11}$$

PROOF

Recall that

$$\Delta(\tau, \eta) = 0 \quad \text{if and only if} \quad \tau = \pm i\Lambda\eta, \ (\tau, \eta) \in \Xi_1,$$

where $\Lambda := (1 - \Gamma^2)^{-1/2}$.

Lemma

When $\tau = \pm i\Lambda\eta$, the eigenvalue ω is purely imaginary.
Each of these roots is simple in the sense that, if $q = \pm\Lambda$, then there exists a neighborhood $\mathscr{V}$ of $(iq\eta, \eta)$ in Ξ_1 and a C^∞–function h_q defined on $\mathscr{V}$ such that

$$\Delta(\tau, \eta) = (\tau - iq\eta) h_q(\tau, \eta), \quad h_q(\tau, \eta) \neq 0 \quad \text{for all } (\tau, \eta) \in \mathscr{V}. \tag{12}$$

Since Ξ_1 is compact, there exists a finite covering $\{\mathscr{V}_1, \ldots, \mathscr{V}_J\}$ of Ξ_1 by such neighborhoods with corresponding matrices $\{T_1, \ldots, T_J\}$, and a smooth **partition of unity** $\{\chi_j(\tau, \eta)\}_{j=1}^J \in C_c^\infty(\mathscr{V}_j)$ such that $\sum_{j=1}^J \chi_j^2 = 1$ on Ξ_1.

Again, define $\Pi_j := \{(\tau,\eta) \in \Xi : \exists\, s > 0,\ s\cdot(\tau,\eta) \in \mathscr{V}_j\}$ and

$$\mathsf{W}(\tau,\eta,y) := \chi_j T_j(\tau,\eta)^{-1}\widehat{u^{\mathrm{nc}}}(\tau,\eta,y), \quad \forall\,(\tau,\eta) \in \Pi_j.$$

Assume that $(\tau,\eta) \in \Pi_j$ and $\operatorname{Re}\tau > 0$. Then $\dfrac{\mathrm{d}\mathsf{W}}{\mathrm{d}y} = T_j^{-1}\mathcal{A}T_j\mathsf{W}$. Hence

$$\frac{\mathrm{d}\mathsf{W}_2}{\mathrm{d}y} = -\omega\mathsf{W}_2, \quad \Longrightarrow\ \mathsf{W}_2 = 0\ \ (\operatorname{Re}\omega < \mathbf{0}).$$

Using the boundary equation (5b), one has

$$\chi_j\widehat{g} = \beta T_j(\tau,\eta)\mathsf{W}(\tau,\eta,0) = \underbrace{\beta E(\tau,\eta)}_{\Delta(\tau,\eta)}\mathsf{W}_1(\tau,\eta,0). \tag{13}$$

- If $\Delta(\tau,\eta)\neq 0$ for all $(\tau,\eta)\in\mathscr{V}_j$, then we proceed as in the previous regular case.
- If $(iq\eta,\eta)\in\mathscr{V}_j$, with $q=\pm\Lambda$, from (12)

$$\Delta(\tau,\eta)=(\tau-iq\eta)h_q(\tau,\eta),\quad h_q(\tau,\eta)\neq 0. \tag{14}$$

Extending $\Delta(\tau,\eta)$ to Π_j as a homogeneous function of degree 1, from (13), (14) we obtain

$$|(\tau-iq\eta)\mathsf{W}_1(\tau,\eta,0)|\lesssim\lambda(\tau,\eta)\big|\chi_j\widehat{g}(\tau,\eta)\big|.$$

Therefore, for all $(\tau,\eta)\in\Pi_j$ with $\gamma=\operatorname{Re}\tau>0$,

$$\gamma\big|\chi_j\widehat{u^{\mathrm{nc}}}(\tau,\eta,0)\big|\lesssim\lambda(\tau,\eta)\big|\chi_j\widehat{g}(\tau,\eta)\big|.$$

Applying Plancherel's theorem yields

$$\gamma\,\|u^{\mathrm{nc}}|_{x_2=0}\|_{L^2_\gamma(\mathbb{R}^2)}\lesssim\|g\|_{H^1_\gamma(\mathbb{R}^2)},$$

that is (11). □

Calculations as in

- 2D compressible vortex sheets, linear stability:
  J.-F. Coulombel–P.S. Indiana Univ. Math. J., 53 (2004), 941–1012,
- 2D compressible elastic flows, linear stability: R.M.Chen–J.Hu–D.Wang, Adv. Math. 311 (2017), 18–60.

The weakly stable case $0 < \Gamma < 1$ (ibvp)

More in general, for the problem

$$\begin{cases} U_{tt} - U_{xx} - U_{yy} = F & \text{for } t \in \mathbb{R},\ x \in \mathbb{R},\ y > 0, \\ \Gamma U_t + U_y = 0 & \text{for } y = 0\,, \\ U = 0 & \text{for } t < 0\,, \end{cases} \tag{15}$$

where F is a given source term such that $F = 0$ for $t < 0$, one can obtain

Theorem

Assume $0 < \Gamma < 1$. For all $m \geq 0$ and for $\gamma \geq 1$, if $u \in H^{m+2}_\gamma(\mathbb{R}^3_+)$ is a solution to (15) the following estimate holds:

$$\gamma \|u\|^2_{H^m_\gamma(\mathbb{R}^3_+)} + \|u^{\mathrm{nc}}|_{x_2=0}\|^2_{H^m_\gamma(\mathbb{R}^2)} \lesssim \|F\|^2_{H^{m+1}_\gamma(\mathbb{R}^3_+)}.$$

$\Longrightarrow$ **Loss of regularity** from the source term.