Euler Equations and Mixed-Type Problems in Gas Dynamics and Geometry

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Dehua Wang

Department of Mathematics, University of Pittsburgh

dhwang@pitt.edu

https://www.pitt.edu/∼dhwang

Outline

Part 1: Brief review on Euler equations and mixed-type problems,

Part 2: Transonic flows past obstacles and in nozzles,

Part 3: Transonic flows in isometric embeddings.

Part 1:

Brief review on Euler equations and mixed-type problems

Gas dynamics and Euler equations

From Wikipedia:

- Gas dynamics is a science in the branch of fluid dynamics, concerned with the study of motion of gases and its effects on physical systems.

- Progress in gas dynamics coincides with the developments of transonic and supersonic flights.

Gas dynamics and shock waves

Shock waves occur in many applications.

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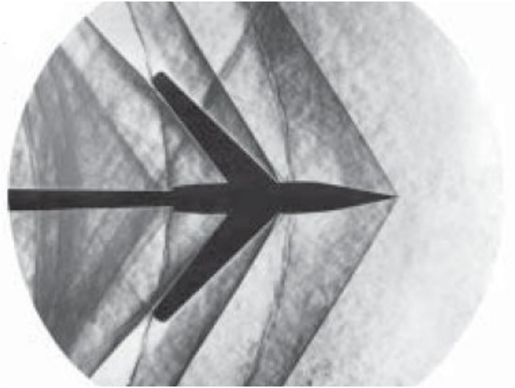
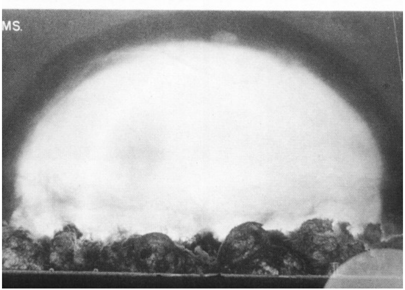
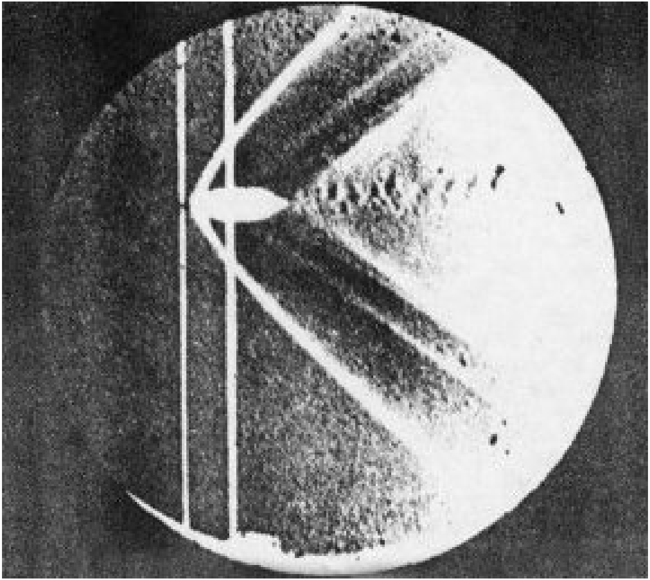


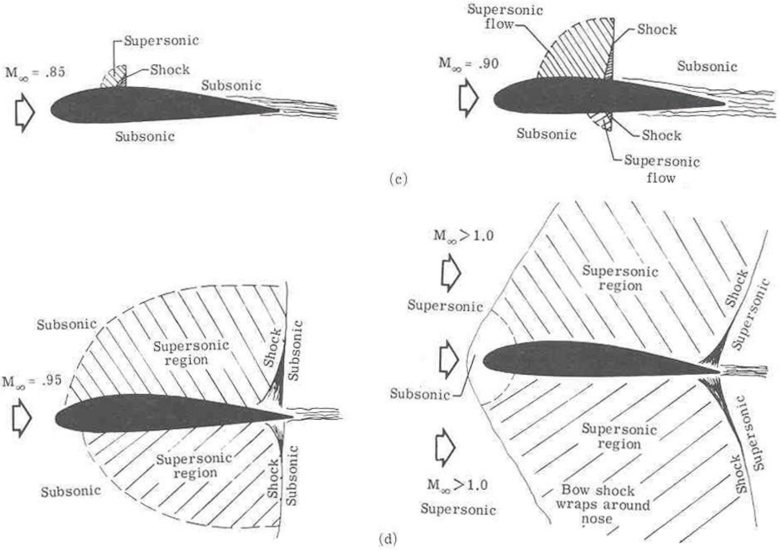
Schlieren photograph of an attached shock on a sharp-nosed supersonic body.



Shock wave propagating into a stationary medium, ahead of the fireball of an explosion.



Shock on a bullet in supersonic flight, published by Ernst Mach in 1887.



Shock on a transonic flow airfoil.

Shock of supernova.

NASA Volcano Image Shows Atmospheric Shockwave.

Compressible Euler Equations

Compressible inviscid fluid flow:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, & \text{(conservation of mass)} \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, & \text{(conservation of momentum)} \\ E_t + \nabla \cdot ((E + p)\mathbf{u}) = 0. & \text{(conservation of energy)} \end{cases}$$

$$E = \frac{1}{2}\rho \mathbf{u}^2 + \rho e, \quad p = p(\rho, e).$$

ρ: density; $\mathbf{u}$: velocity; p: pressure;
E: total energy; e: internal energy.

Other variables:
θ: temperature; S: entropy; $\tau = \frac{1}{\rho}$: special volume.

First Law of Thermodynamics:

$$\theta dS = de + p d\tau = de - \frac{p}{\rho^2} d\rho.$$

For a polytropic gas,

$$p = R\rho\theta, \qquad e = c_v\theta, \qquad \gamma = 1 + \frac{R}{c_v},$$

$$p = p(\rho, S) = \kappa\rho^\gamma e^{S/c_v}, \qquad e = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1} e^{S/c_v},$$

$R > 0$: the universal gas constant divided by the effective molecular weight of the particular gas;
$c_v > 0$: the specific heat at constant volume;
$\gamma > 1$: the adiabatic exponent; $\kappa > 0$: constant under scaling.

For smooth solutions, the entropy $S(\rho, E)$ is conserved along fluid particle trajectories:

$$\partial_t(\rho S) + \nabla \cdot (\mathbf{m} S) = 0.$$

Isentropic flow:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0. \end{cases}$$

$$p = \frac{\rho^\gamma}{\gamma}, \quad \gamma > 1.$$

Elastodynamic equations:

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \mathrm{div}\,(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \mathrm{div}(\rho\, \mathbf{F}\,\mathbf{F}^\top), \\ \mathbf{F}_t + \mathbf{u} \cdot \nabla \mathbf{F} = \nabla \mathbf{u}\, \mathbf{F}. \end{cases}$$

Magnetohydrodynamic equations:

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \mathrm{div}\,(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H}, \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = 0, \quad \mathrm{div} \mathbf{H} = 0. \end{cases}$$

The general conservation laws:

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \qquad \mathbf{u} \in \mathbb{R}^n, \quad \mathbf{x} \in \mathbb{R}^d, \tag{1}$$

where $\mathbf{f} = (\mathbf{f}_1, \cdots, \mathbf{f}_d) : \mathbb{R}^n \to (\mathbb{R}^n)^d$ is a nonlinear mapping with $\mathbf{f}_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \cdots, d$.

The **hyperbolicity** of system (1) requires that, for any $\omega \in S^{d-1}$, the matrix $(\nabla \mathbf{f}(\mathbf{u}) \cdot \omega)_{n \times n}$ have n real eigenvalues $\lambda_i(\mathbf{u}, \omega), i = 1, 2, \cdots, n$, and be diagonalizable.

For the one-dimensional isentropic Euler equations of gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p \right) = 0, \end{cases}$$

for $x \in \mathbb{R}$ and $t > 0$, $m = \rho u$, with the γ-law for pressure:

$$p(\rho) = \rho^\gamma / \gamma, \quad \gamma > 1. \tag{2}$$

For the case $1 < \gamma \leq 3$, which is of physical significance, the eigenvalues are

$$\lambda_1 = u - c, \quad \lambda_2 = u + c,$$

where $c = \rho^\theta$, with $\theta = \frac{\gamma - 1}{2} \in (0, 1]$, is the sound speed.

Strictly hyperbolic if $\rho > 0$.

A function $\eta : \mathfrak{D} \to \mathbb{R}$ is called an **entropy** of system (1) if there exists a vector function $\mathbf{q} : \mathfrak{D} \to \mathbb{R}^d, \mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_d)$, satisfying

$$\nabla \mathbf{q}_i(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_i(\mathbf{u}), \qquad i = 1, \ldots, d.$$

An entropy $\eta(\mathbf{u})$ is called a convex entropy in $\mathfrak{D}$ if

$$\nabla^2 \eta(\mathbf{u}) \geq 0 \qquad \text{for any } \mathbf{u} \in \mathfrak{D}$$

and a strictly convex entropy in $\mathfrak{D}$ if $\nabla^2 \eta(\mathbf{u}) \geq c_0 I$.

The entropy condition:

$$\partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \leq 0$$

in the sense of distributions for any C^2 convex entropy-entropy flux pair $(\eta, \mathbf{q})$.

The relative entropy and entropy flux pair:

$$\begin{aligned}
\alpha(\mathbf{u}, \mathbf{v}) &= \eta(\mathbf{u}) - \eta(\mathbf{v}) - \nabla\eta(\mathbf{v})(\mathbf{u} - \mathbf{v}), \\
\beta(\mathbf{u}, \mathbf{v}) &= \mathbf{q}(\mathbf{u}) - \mathbf{q}(\mathbf{v}) - \nabla\eta(\mathbf{v})(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v}))
\end{aligned}$$

satisfies

$$\begin{aligned}
&\partial_t \alpha(\mathbf{u}, \mathbf{v}) + \nabla_{\mathbf{x}} \cdot \beta(\mathbf{u}, \mathbf{v}) \\
&\leq -\{\partial_t(\nabla\eta(\mathbf{v}))(\mathbf{u} - \mathbf{v}) + \nabla_{\mathbf{x}}(\nabla\eta(\mathbf{v}))(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v}))\}.
\end{aligned}$$

The system (1) is called **symmetrizable**, if there is a positive definite symmetric matrix $A_0(\mathbf{u})$, such that $A_i(\mathbf{u}) = A_0(\mathbf{u})\nabla \mathbf{f}_i(\mathbf{u})$ is symmetric. The matrix $A_0(\mathbf{u})$ is called the symmetrizing matrix. Denote $A(\mathbf{u}) = (A_1(\mathbf{u}), \cdots, A_d(\mathbf{u}))$, then

$$A_0(\mathbf{u})\partial_t \mathbf{u} + A(\mathbf{u})\nabla \mathbf{u} = 0.$$

Theorem

A system in (1) *endowed with a strictly convex entropy* η *in a state domain* $\mathfrak{D}$ *must be symmetrizable and hence hyperbolic in* $\mathfrak{D}$.

For the isentropic Euler equations, the mechanical energy and energy flux

$$\eta_* = \frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \frac{\rho^\gamma}{\gamma(\gamma-1)}, \quad \text{with } \mathbf{m} = \rho\mathbf{u},$$

$$\mathbf{q}_* = \frac{\mathbf{m}}{\rho}\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \frac{\rho^\gamma}{\gamma-1}\right)$$

is a strictly convex entropy-entropy flux pair when $\rho > 0$.

The Euler system is a symmetrizable hyperbolic system.

Local Existence of Smooth Solution

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \end{cases}$$

Cauchy problem for $U = (\rho, \mathbf{u})$:

$$U|_{t=0} = U_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3.$$

Theorem (Local existence)

For

$$U_0 \in H^s \cap L^\infty(\mathbb{R}^3), \quad s > 5/2, \quad \rho_0(\mathbf{x}) > 0,$$

$\exists$ *a finite time* $T \in (0, \infty)$, *s.t. the Cauchy problem has a unique smooth solution* $U \in C^1 \cap L^\infty(\mathbb{R}^3 \times [0, T])$, $\rho(\mathbf{x}, t) > 0$, *and* $U \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$.

- *Friedrichs, Lax, Li-Yu; Kato, Majda, Makino-Ukai-Kawashima, Q. Wang, ...*

Formation of Singularities

Cauchy problem with smooth initial data:

$$(\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0)(\mathbf{x}),\ \rho_0 > 0; \quad (\rho_0, \mathbf{u}_0)(\mathbf{x}) = (\bar{\rho}, 0),\ \text{for } |\mathbf{x}| \geq R.$$

Finite propagation speed: $\sigma = \sqrt{p_\rho(\bar{\rho})}$ (sound speed),

$$(\rho, \mathbf{u})(\mathbf{x}, t) = (\bar{\rho}, 0), \quad \text{if} \quad |\mathbf{x}| \geq R + \sigma t.$$

$$P(t) = \int_{\mathbb{R}^3} \left(p(\mathbf{x}, t)^{1/\gamma} - \bar{p}^{1/\gamma}\right) d\mathbf{x},\ \bar{p} = p(\bar{\rho}),$$
$$F(t) = \int_{\mathbb{R}^3} \mathbf{x} \cdot \rho\mathbf{u}(\mathbf{x}, t) d\mathbf{x}.$$

Theorem (*Sideris, 1985*)

If $(\rho, \mathbf{u})(\mathbf{x}, t)$ *is a* C^1 *solution for* $0 < t < T$*, and*

$$P(0) \geq 0, \qquad F(0) > \alpha\sigma R^4 \max_{\mathbf{x}} \rho_0(\mathbf{x}), \quad \alpha = 16\pi/3,$$

then the lifespan T *of the* C^1 *solution is finite.*

- Formation of Singularities:

  Lax, John, Liu; Klainerman-Majda, Sideris, Rammaha, Hu-W., Christodoulou-Miao, Luk-Speck, An-Chen-Yin, Buckmaster-Shkoller-Vicol, ...

- Formation of shocks for 2D isentropic compressible Euler (Buckmaster-Shkoller-Vicol, 2022)

  (Rough statement) For an open set of smooth initial data with $O(1)$ amplitude and with minimum initial slope given at initial time t_0 to equal $-1/\varepsilon$, for $\varepsilon > 0$ taken sufficiently small, there exist smooth solutions of the Euler equations with $O(1)$ vorticity, which form an asymptotically self-similar shock in finite time T_*, such that $T_* - t_0 = O(1)$.

Isentropic Euler Equations: weak solutions

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \end{cases}$$

1-D Problem:

- Small BV solution: Glimm scheme, wave-front tracking, vanishing viscosity;

  Glimm, Glimm-Lax, Liu, Dafermos, Bressen, Liu-Yang, Bianchini-Bressan, Vasseur, and many others ......

- Large L^∞ solution: vanishing viscosity, finite difference, kinetic formulation, via compensated compactness methods.

  DiPerna, Ding-Chen-Luo, Chen, Lions-Perthame-Souganidis-Tadmor, Chen-LeFloch, Huang-Wang ($\gamma = 1$), ...

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  ......

Other studies: surveyed in Dafermos' book.
Hsiao-Zhang, Slemrod, Smoller, Nishida, Chen-LeFloch, Pan, Huang, Serre, Luo, Jessen, Liu-Yang, Goodman-Xin, Temple-Young, Li, Greenberg-Rascle, Chen-Wang, Tzavaras, Jin-Xin, Wang, Keyfitz, Chen-Frid, Shearer, Lewicka, Christoforou, Trivisa, Holden-Risebro, Zumbrunn, Dafermos-Pan, Huang-Pan, LeFloch-Westdickenberg, Gangbo-Westdickenberg, De Lellis-Szekelyhidi, ...

M-D Problem

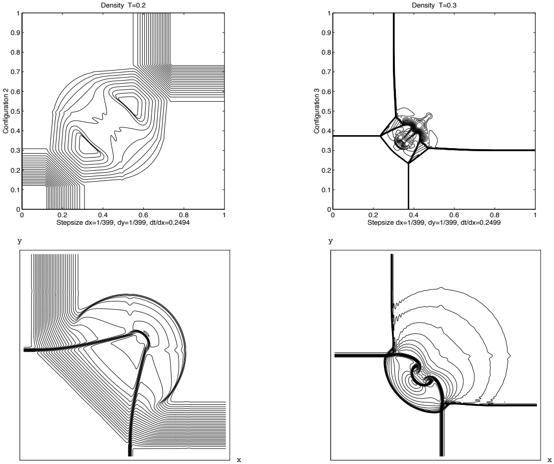
Very difficult: 1-D methods do not work.

Lots of progress recently:

Morawetz, Gamba-Morawetz, Canic-Keyfitz-Lieberman-Kim-Jedgic, Chen-Feldman, Zheng, Li-Zheng, Serre, Xin-Yin, Hunter, Zhang, S.-X. Chen, Liu-Elling, Chen-Dafermos-Slemrod-Wang, Xin-Xie, Chen-Slemrod-Wang, Chen-Wang-Yang, LeFloch-Westdickenberg, Luo-Smoller, Gangbo-Westdickenberg, Bae-Chen-Feldman, ...

Convex integration: *De Lellis-Szekelyhidi, Chen-Vasseur-Yu, ...*

Complex Structures of 2-D Riemann Solutions



Lax-Liu, Kurganov-Tadmor

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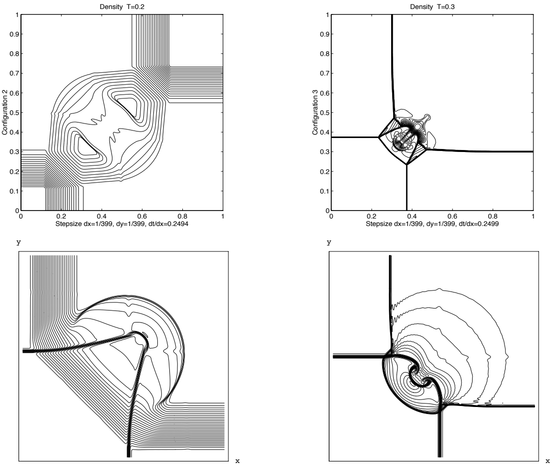


Lax-Liu, Kurganov-Tadmor

Recent work: Chen-Cliffe-Huang-Liu-Wang, 2023.

M-D compressible flows: mixed-type, free boundary

Riemann problems,

self-similar solutions,

shock reflections,

transonic flows,

vortex sheets and interfaces,

......

The general linear second-order equation with two independent variables for $u(x, y)$:

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g,$$

where a, b, c, d, e, f, g are given functions of (x, y).

The characteristic equation:

$$a\lambda^2 - 2b\lambda + c = 0.$$

The eigenvalues:

$$\lambda = \frac{b \pm \sqrt{b^2 - ac}}{a}.$$

The equation is *hyperbolic* if the characteristic equation has two real distinct eigenvalues ($b^2 - ac > 0$), is *elliptic* if it has no real eigenvalues ($b^2 - ac < 0$), and is *parabolic* if it has one real eigenvalues ($b^2 - ac = 0$).

Or, it is called elliptic if all eigenvalues of the matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

have the same sign, parabolic if A is singular, and hyperbolic if the two eigenvalues of A have the opposite signs.

Basic equations of three types

Laplace equation: elliptic

$$\Delta u = 0.$$

Heat equation: parabolic

$$u_t - \Delta u = 0.$$

Wave equation: hyperbolic

$$u_{tt} - \Delta u = 0.$$

Equations of mixed types: with fixed boundary

The Tricomi equation

$$u_{xx} - xu_{yy} = 0.$$

The Keldysh equation: (also called the Cinquini-Cibrario's equation sometimes)

$$u_{xx} - yu_{yy} = 0.$$

Lavrentyev-Bitsadze equation:

$$u_{xx} + \sin(x)u_{yy} = 0.$$

Equations of mixed types: with free boundary

• The equation:

$$u_{xx} - uu_{yy} = 0,$$

is hyperbolic if $u > 0$, elliptic if $u < 0$, and parabolic if $u = 0$. So it is of mixed type, and the boundary $u = 0$ separating the hyperbolic and elliptic parts is a free boundary.

- The equation for a two-dimensional steady potential flow is:

$$(c^2-u^2)\varphi_{xx}-2uv\varphi_{xy}+(c^2-v^2)\varphi_{yy}=0,$$

where $(u,v)=\nabla\varphi=(\varphi_x,\varphi_y)$, φ is the velocity potential, and c is the sound speed given by the Bernoulli's law:

$$c^2=1-\frac{\gamma-1}{2}\left(u^2+v^2\right),$$

with $\gamma>1$ constant. The characteristic equation is

$$(c^2-u^2)\lambda^2+2uv\lambda+(c^2-v^2)=0,$$

with eigenvalues

$$\lambda=\frac{-uv\pm c\sqrt{u^2+v^2-c^2}}{c^2-u^2}.$$

Thus the equation is hyperbolic if $u^2+v^2>c^2$ (i.e., supersonic), elliptic if $u^2+v^2<c^2$ (i.e., subsonic), and parabolic if $u^2+v^2=c^2$ (i.e., sonic). It is of mixed type, and the sonic curve $u^2+v^2=c^2$ is a free boundary.

Chen-Feldman 2018 (Research Monograph): The Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures, 832 pages, Annals of Mathematics Studies, 197, Princeton University Press, 2018.

Mathematical Challenges of Mixed-Type PDEs

- The transition boundary between the elliptic and hyperbolic phases is a priori unknown, thus most of the classical approaches do not work.
- New approaches are needed to deal with the free boundary problems, including optimal estimates of solutions to nonlinear degenerate PDEs, corner singularities, ...

Part 2:

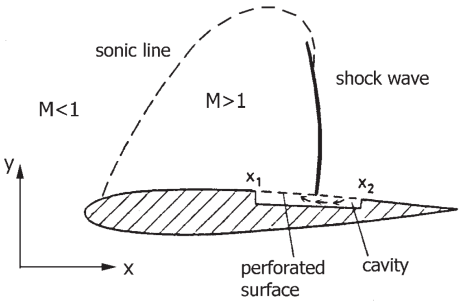
Transonic flows past obstacles and in nozzles

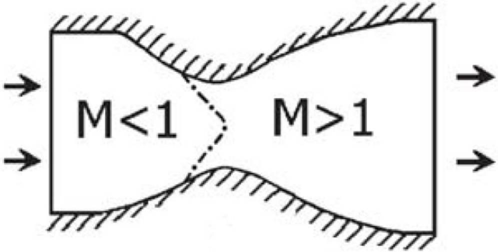
Transonic flows in gas dynamics

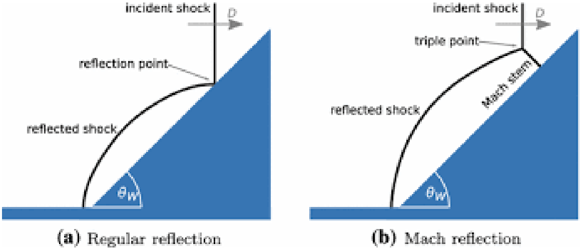
Transonic flows occur in gas dynamics, astronomy, astrophysics, and so on.

- Transonic flow is where air flows above, at, and below the speed of sound at the same time at different points on an object.
- Supersonic flow, sonic flow, subsonic flow.
- Singularities: shock wave, rarefaction wave, contact discontinuity, ...

M=Mach number=$\frac{\text{flow speed}}{\text{sound speed}}$ $M<1$: elliptic, $M>1$: hyperbolic.







Lau-Chapdelaine, S.SM., Radulescu, M.I. Non-uniqueness of solutions in asymptotically self-similar shock reflections. Shock Waves 23, 595-602 (2013).

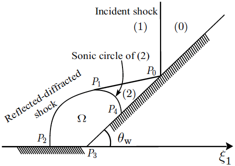


FIGURE 3. Supersonic regular reflection-diffraction configuration [8].

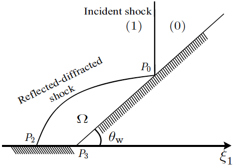
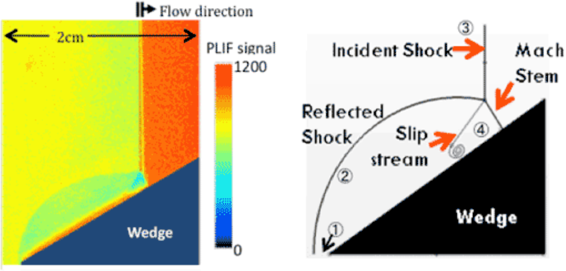


FIGURE 4. Subsonic regular reflection-diffraction configuration [8].

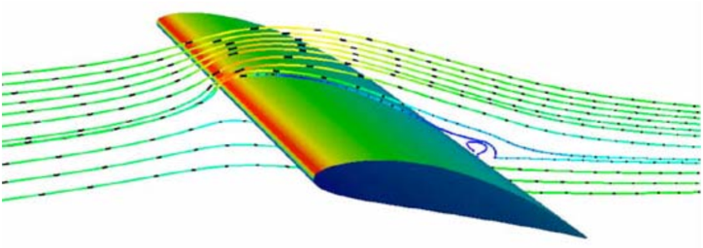
G.-Q. Chen, Morawetz's contributions to the mathematical theory of transonic flows, shock waves, and partial differential equations of mixed type. Bull. Amer. Math. Soc. (N.S.)61(2024), no.1, 161-171.

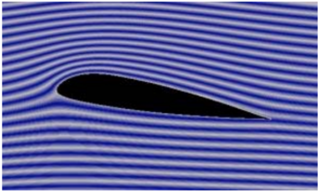
G.-Q. Chen, M. Feldman, The mathematics of shock reflection-diffraction and von Neumann's conjectures. Annals of Mathematics Studies, vol. 197, Princeton University Press, Princeton, NJ, 2018.

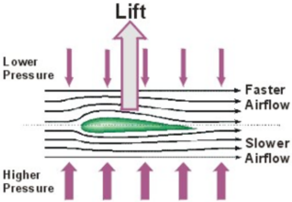


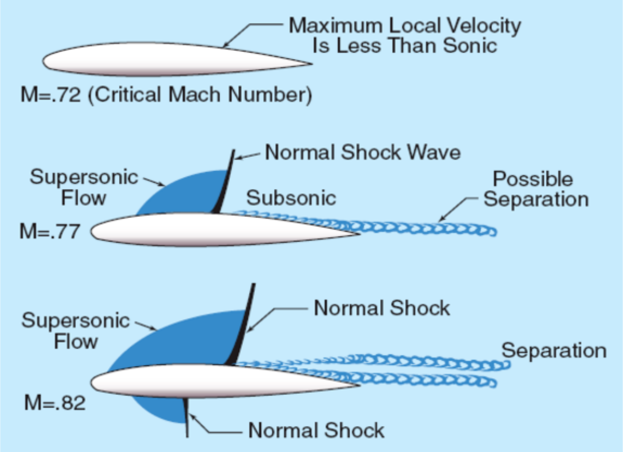
https://psaap.stanford.edu/heat_release_modeling/temperature_imaging.html

Transonic flow past an airfoil

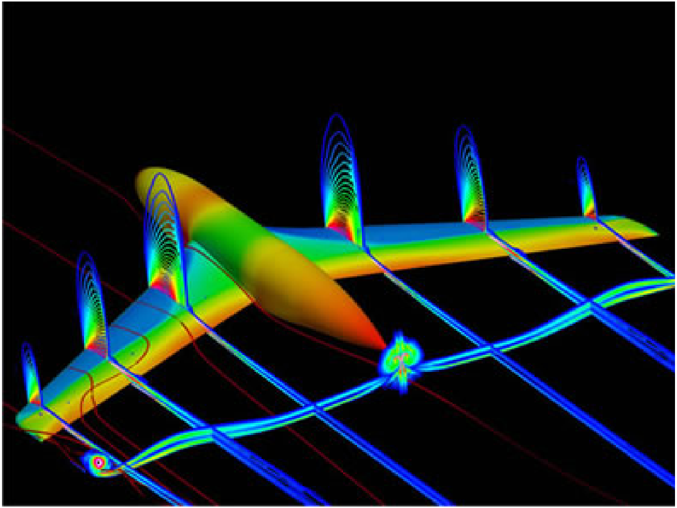





Lift
Lower Pressure
Faster Airflow
Slower Airflow
Higher Pressure



From Airplane Flying Handbook



2-D Euler Equations for Steady Irrotational Flows

$$\begin{cases} v_x - u_y = 0, \\ (\rho u)_x + (\rho v)_y = 0, \\ (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho uv)_x + (\rho v^2 + p)_y = 0, \end{cases} \qquad p = p(\rho) = \rho^\gamma/\gamma,\ \gamma \geq 1.$$

Bernoulli's law:

$$\rho = \left(1 - \frac{\gamma - 1}{2} q^2\right)^{\frac{1}{\gamma-1}}, \ \textit{or} \quad q^2 - q_{cr}^2 = \frac{2}{\gamma + 1}\left(q^2 - c^2\right),$$

where $q^2 = u^2 + v^2, \quad c^2 = p'(\rho) = 1 - \frac{\gamma-1}{2} q^2,$

$$q_{cr} \equiv \sqrt{\frac{2}{\gamma+1}}, \quad q \leq q_{cav} \equiv \sqrt{\frac{2}{\gamma-1}}.$$

2-D Euler Equations for Steady Irrotational Flows

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$$q_{cr} \equiv \sqrt{\frac{2}{\gamma+1}}, \quad q \leq q_{cav} \equiv \sqrt{\frac{2}{\gamma-1}}.$$

The flow is subsonic if $q < q_{cr}$, sonic if $q = q_{cr}$,

and supersonic if $q > q_{cr}$.

Subsonic Flow

Existence of subsonic solutions: *Bers, Shiffman (50's)*

For a given $w_\infty = (u_\infty, v_\infty)$, there exists $\hat{q} < q_{cr}$, s.t. the problem has a unique subsonic solution (u, v) for $q_\infty := |w_\infty| < \hat{q}$. The maximum speed $q_m \to q_{cr}$ as $q_\infty \to \hat{q}$.

Bers, Shiffman, Serrin, Finn, Gilbarg, Dong, J. Chen, C. Wang-Xie-Xin, ...

Subsonic-Sonic Flow

$q_m \to q_{cr}$ as $q_\infty \nearrow \hat{q}$: sonic points appear.

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Existence of sonic-subsonic solutions:

Chen-Dafermos-Slemrod-D.W. (CMP)

Let $q_\infty^\varepsilon < \hat{q}$ be a sequence of speeds at ∞, and let $(u^\varepsilon, v^\varepsilon)$ be the corresponding subsonic solutions, then, as $q_\infty^\varepsilon \nearrow \hat{q}$, the sequence $(u^\varepsilon, v^\varepsilon)$ possesses a subsequence that converges strongly to a weak solution (u, v) with $q = |(u, v)| \le q_{cr}$.

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Approach: compensated compactness, momentum equations.

Compensated Compactness

Recall: for a sequence $u_k : \Omega \to \mathbb{R}^m$ bounded in $L^\infty(\Omega)$, there exists a subsequence (still denoted) u_k and a function $u \in L^\infty(\Omega)$ such that $u_k \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$, i.e.,

$$\int_\Omega u_k g dx \to \int_\Omega u g dx, \quad \text{as } k \to \infty,$$

for each $g \in L^1(\Omega)$.

For a continuous function $f \in C(\mathbb{R}^m)$, $f(u_k)$ is bounded in $L^\infty(\Omega; \mathbb{R}^m)$ and thus $f(u_k) \overset{*}{\rightharpoonup} \bar{f}$. The question is:

$$\bar{f} = f(u)?$$

The answer is no, in general.

A counterexample:

Take $u_k = \sin kx$, $\phi \in C_0^1(\mathbb{R})$ a test function with compact support. Then

$$\int_{\mathbb{R}} \sin kx\, \phi(x) dx = \frac{1}{k} \int_{\mathbb{R}} \cos kx\, \phi'(x) dx \to 0, \text{ as } k \to \infty$$

since $\int_{\mathbb{R}} \cos kx\, \phi'(x) dx$ is bounded; but, for

$$u_k^2 = \sin^2 kx = \frac{1}{2}(1 - \cos 2kx),$$

$$\int_{\mathbb{R}} \sin^2 kx \phi(x) dx = \frac{1}{2} \int_{\mathbb{R}} \phi(x) dx - \frac{1}{2} \int_{\mathbb{R}} \cos 2kx \phi(x) dx \to \frac{1}{2} \int_{\mathbb{R}} \phi(x) dx.$$

Thus,

$$u_k = \sin kx \stackrel{*}{\rightharpoonup} u = 0, \quad u_k^2 = \sin^2 kx \stackrel{*}{\rightharpoonup} \frac{1}{2} \neq u^2 = 0.$$

Lemma (Tartar) Suppose that $v^\varepsilon : \mathbb{R}^2_+ = \mathbb{R} \times [0, \infty) \to \mathbb{R}^m$ is a sequence of uniformly bounded measurable functions, i.e.,

$$v^\varepsilon(x, t) \in K, \quad a.e.$$

for a bounded set $K \in \mathbb{R}^m$, and that, for two function pairs (η_i, q_i), $i = 1, 2$,

$$\eta_i(v^\varepsilon)_t + q_i(v^\varepsilon)_x \quad \text{is compact in} \quad H^{-1}_{loc}.$$

Then there exists a subsequence (still labeled v^ε) and Young measures

$$\nu_{x,t} : \mathbb{R}^2_+ \to \mathrm{Prob}(\mathbb{R}^m), \quad \mathrm{supp}\, \nu_{x,t} \subset \overline{K},$$

such that

(1). For any continuous function f, the weak limit has the following Young measure representation,

$$w^*\text{-}\lim f(v^\varepsilon) = \langle \nu_{x,t}(\lambda), f(\lambda)\rangle = \int_{\mathbb{R}^m} f(\lambda) d\nu_{x,t}(\lambda),$$

and the Young measure $\nu_{x,t}$ commutes with the 2×2 determinant mapping acting on the function pairs, that is, the following commutativity relation holds,

$$\langle \nu_{x,t}, \eta_1 q_2 - \eta_2 q_1\rangle = \langle \nu_{x,t}, \eta_1\rangle\langle \nu_{x,t}, q_2\rangle - \langle \nu_{x,t}, \eta_2\rangle\langle \nu_{x,t}, q_1\rangle.$$

(2). $v^\varepsilon(x,t) \to v(x,t)$ strongly if and only if $\nu_{x,t}$ is a Dirac mass, i.e.,

$$\nu_{x,t} = \delta_{u(x,t)}, \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^2_+.$$

Compensated Compactness for Subsonic-Sonic Flow

$w^\varepsilon(x,y) = (u^\varepsilon, v^\varepsilon)(x,y)$, $(x,y) \in \Omega \subset \mathbb{R}^2$:

(1) $q^\varepsilon(x,y) = |w^\varepsilon(x,y)| \le q_{cr}$ a.e. in Ω;
(2) $\partial_x \eta_k(w^\varepsilon) + \partial_y q_k(w^\varepsilon), k = 1, 2$, are compact in $H^{-1}_{loc}(\Omega)$,

where $\quad (\eta_1, q_1) = (\rho u^2 + p,\ \rho u v), \quad (\eta_2, q_2) = (\rho u v,\ \rho v^2 + p).$

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where $(\eta_1, q_1) = (\rho u^2 + p,\ \rho uv)$, $(\eta_2, q_2) = (\rho uv,\ \rho v^2 + p)$.

Then, the div-curl lemma (Tartar-Murat) implies the commutation identity:

$$
\begin{aligned}
&\langle \nu(w), \eta_i(w) q_j(w) - q_i(w)\eta_j(w)\rangle \\
&\quad = \langle \nu(w), \eta_i(w)\rangle\langle \nu(w), q_j(w)\rangle - \langle \nu(w), q_i(w)\rangle\langle \nu(w), \eta_j(w)\rangle,
\end{aligned}
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where $\nu = \nu_{x,y}(w)$ is the associated Young measure (probability measure) for the sequence $w^\varepsilon(x,y)$.

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where $\nu = \nu_{x,y}(w)$ is the associated Young measure (probability measure) for the sequence $w^\varepsilon(x,y)$.

Claim: ν is a Dirac measure.

Proof of Convergence

$$\langle \nu(w_1) \otimes \nu(w_2),\ I(w_1, w_2) \rangle = 0,$$

where

$$
\begin{aligned}
& I(w_1, w_2) \\
& = \big(\eta_1(w_1) - \eta_1(w_2)\big)\big(q_2(w_1) - q_2(w_2)\big) \\
& \quad - \big(q_1(w_1) - q_1(w_2)\big)\big(\eta_2(w_1) - \eta_2(w_2)\big) \\
& = -\rho_1 \rho_2 (u_1 v_2 - u_2 v_1)^2 - \frac{\gamma + 1}{\gamma - 1}(p_1 - p_2)^2 \frac{q_{cr}^2 - \tilde{q}^2}{\frac{2}{\gamma - 1} - \tilde{q}^2} \\
& \leq 0,
\end{aligned}
$$

where $\tilde{q} \leq q_{cr}$ is between q_1 and q_2.

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&\quad - \big(q_1(w_1) - q_1(w_2)\big)\big(\eta_2(w_1) - \eta_2(w_2)\big) \\
&= -\rho_1\rho_2(u_1 v_2 - u_2 v_1)^2 - \frac{\gamma+1}{\gamma-1}(p_1 - p_2)^2 \frac{q_{cr}^2 - \tilde{q}^2}{\frac{2}{\gamma-1} - \tilde{q}^2} \\
&\leq 0,
\end{aligned}$$

where $\tilde{q} \leq q_{cr}$ is between q_1 and q_2.

- *Extension to higher-dimensions, full Euler equations, or other related problems:*

  F.-M. Huang-T. Wang-Y. Wang, C. Wang-Xie-Xin, ...

Transonic Flow

$q_\infty > \hat{q}$: transonic flow.

Transonic Flow

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L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, 1958, pp. 3, & 135:

> These (transonic flow) problems, while admittedly difficult, are exceedingly challenging and give is a glimpse of the long lost golden age of the unity of science. Indeed, physicists interested in them demand rigorous mathematical proofs, ...
>
> It is hardly necessary to point out how interesting it would be to obtain general existence theorems and effective methods of computation for the type of flow considered here in the case where the profile is an arbitrarily given curve. This problem is probably rather different.

Courant-Friedrichs, *Supersonic Flow and Shock Waves*, 1962, pp 367:

> ... even then a rigorous proof seems beyond the present possibilities of analysis, ...

Morawetz's work

Morawetz: 1985, 1995, 2004

If the viscous approximation problem

$$\begin{cases} v_x - u_y = R_1, \\ (\rho u)_x + (\rho v)_y = R_2, \end{cases}$$

with Bernoulli's law: $\rho = \rho(q) = \left(1 - \frac{\gamma-1}{2} q^2\right)^{\frac{1}{\gamma-1}}$,
(where R_1 and R_2 are the artificial viscosity terms to be determined,) yields approximate solutions satisfying the compensated compactness framework, then the viscous solutions converge to a solution of the transonic flow problem.

?? Viscous problem ??

An Effective Viscous Problem

Chen-Slemrod-D.W. (ARMA)

Polar coordinates in the phase plane:

$$u = q\cos\theta, \qquad v = q\sin\theta.$$

The viscous problem:

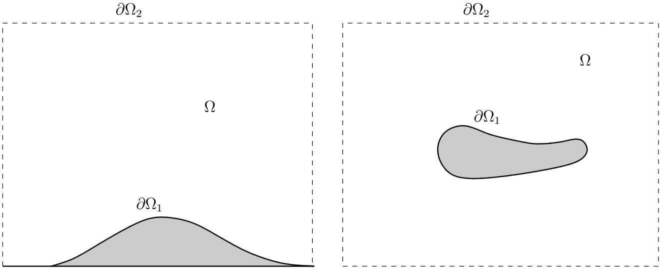
$$\begin{cases} v_x - u_y = R_1 = \varepsilon\Delta\theta, \\ (\rho u)_x + (\rho v)_y = R_2 = \varepsilon\nabla\cdot(\sigma_2(\rho)\nabla\rho), \end{cases}$$

where σ_2 is positive, smooth, bounded, satisfying

$$\sigma_2 = 1 - \frac{c^2}{q^2} \qquad \text{for } q > \frac{2}{\sqrt{3-\gamma}}c > c \ (q > \sqrt{2}q_{cr}),$$

$$1 \le \gamma < 3, \qquad c^2 = p'(\rho) = 1 - \frac{\gamma-1}{2}q^2.$$

Boundary Conditions



(a) (b)

$$
\begin{cases}
\nabla\theta \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega_1, \\
\varepsilon\sigma_2 \nabla\rho \cdot \mathbf{n} = -|\rho\,(u, v) \cdot \mathbf{n}| \qquad \text{on } \partial\Omega_1, \\
(u, v) - (u_\infty, v_\infty) = 0 \qquad \text{on } \partial\Omega_2 \text{ with } q_\infty < q_{cav},
\end{cases}
$$

Riemann Invariants

$$\begin{bmatrix} -\sin\theta & -q\cos\theta \\ \frac{c^2-q^2}{c^2q}\cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} q \\ \theta \end{bmatrix}_x + \begin{bmatrix} \cos\theta & -q\sin\theta \\ \frac{c^2-q^2}{c^2q}\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} q \\ \theta \end{bmatrix}_y = \begin{bmatrix} -R_1 \\ \frac{1}{\rho q}R_2 \end{bmatrix}.$$

Eigenvalues and left eigenvectors:

$$\lambda_\pm = -\sin\theta \pm \frac{\sqrt{q^2-c^2}}{c}\cos\theta, \quad \mu_\pm = \cos\theta \pm \frac{\sqrt{q^2-c^2}}{c}\sin\theta; \quad \left(\mp\frac{\sqrt{q^2-c^2}}{qc},\ 1\right).$$

The Riemann invariants $W_\pm$:

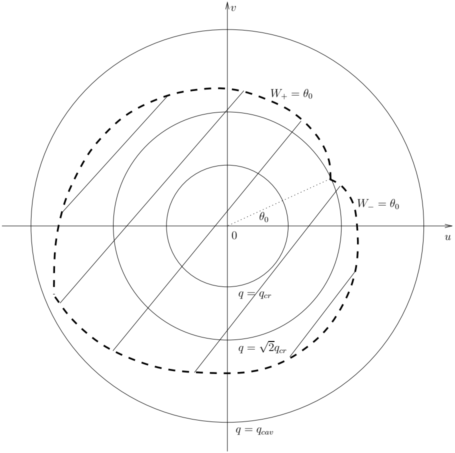
$$\frac{\partial W_\pm}{\partial\theta} = 1, \quad \frac{\partial W_\pm}{\partial q} = \mp\frac{\sqrt{q^2-c^2}}{qc} \qquad \text{for } q \geq c,$$

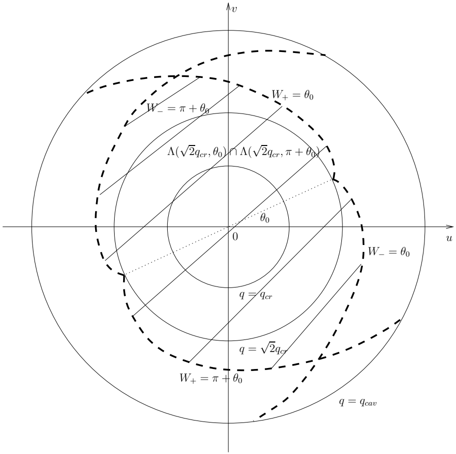
satisfy

$$\lambda_\pm \frac{\partial W_\pm}{\partial x} + \mu_\pm \frac{\partial W_\pm}{\partial y} = -\frac{\partial W_\pm}{\partial q}R_1 + \frac{1}{\rho q}\frac{\partial W_\pm}{\partial\theta}R_2.$$

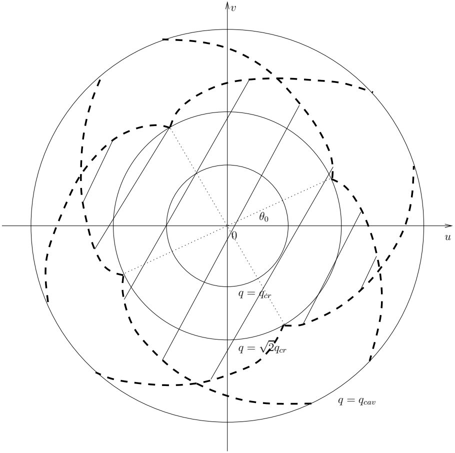
Invariant Regions

$1 \leq \gamma < 3$





v
$W_+ = \theta_0$
$W_- = \pi + \theta_0$
$\Lambda(\sqrt{2}q_{cr}, \theta_0) \cap \Lambda(\sqrt{2}q_{cr}, \pi + \theta_0)$
θ_0
0
u
$W_- = \theta_0$
$q = q_{cr}$
$q = \sqrt{2}q_{cr}$
$W_+ = \pi + \theta_0$
$q = q_{cav}$



v
u
θ_0
0
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$q = \sqrt{2}q_{cr}$
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Compensated Compactness and Convergence

$w^\varepsilon(x,y) = (u^\varepsilon, v^\varepsilon)(x,y)$, $(x,y) \in \Omega \subset \mathbb{R}^2$:

(1) $q^\varepsilon(x,y) = |w^\varepsilon(x,y)| \le q_*$ a.e. in Ω, for some positive constant $q_* < q_{cav} < \infty$;

(2) $\partial_x Q_{1\pm}(w^\varepsilon) + \partial_y Q_{2\pm}(w^\varepsilon)$ are compact in $H^{-1}_{loc}(\Omega)$, for the entropy-entropy flux pairs (Q_1, Q_2),

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$$
\begin{aligned}
&\langle \nu(w), Q_{1+}(w) Q_{2-}(w) - Q_{1-}(w) Q_{2+}(w) \rangle \\
&= \langle \nu(w), Q_{1+}(w) \rangle \langle \nu(w), Q_{2-}(w) \rangle - \langle \nu(w), Q_{1-}(w) \rangle \langle \nu(w), Q_{2+}(w) \rangle,
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where $\nu = \nu_{x,y}(w)$, $w = (u,v)$, is the Young measures for $w^\varepsilon(x,y)$,

$$
\langle \nu(w) \otimes \nu(w'),\ I(w,w')\rangle = 0,
$$

$$
\begin{aligned}
I(w,w') =& (Q_{1+}(w) - Q_{1+}(w'))(Q_{2-}(w) - Q_{2-}(w')) \\
& - (Q_{2+}(w) - Q_{2+}(w'))(Q_{1-}(w) - Q_{1-}(w')),
\end{aligned}
$$

ν is a Dirac measure.

The Entropy-Entropy Flux Pairs (Q_1, Q_2)

$$Q_{1x} + Q_{2y} = -V_\theta R_1 + \frac{q^2}{c^2 - q^2} V_\rho R_2,$$

with

$$\frac{c^2}{\rho q} V_{\theta\theta} + \left(\frac{q^2}{c^2 - q^2} V_\rho \right)_\rho = 0.$$

Generators H: $(\mu'(\rho) = c^2/q^2)$

$$\rho H_{\mu\theta} - H_\theta = -V_\theta, \qquad H_\mu + \frac{1}{\rho} H_{\theta\theta} = \frac{q^2}{c^2 - q^2} V_\rho,$$

satisfying the generalized Tricomi equation:

$$H_{\mu\mu} + \frac{1}{\rho^2}(1 - M^2) H_{\theta\theta} = 0, \qquad M = q/c,$$

The Loewner-Morawetz relation:

$$Q_1 = \rho q H_\mu \cos\theta - q H_\theta \sin\theta, \qquad Q_2 = \rho q H_\mu \sin\theta + q H_\theta \cos\theta.$$

Existence of Transonic Solution:

Chen-Slemrod-D.W.

Let $v_\infty = 0$, $|u_\infty| < q_{cav}$, and $1 \leq \gamma < 3$. Assume $q^\varepsilon(x, y) \geq \alpha(\delta) > 0$ for any $(x, y) \in \Omega_\delta = \{(x, y) \in \Omega : \text{dist}((x, y), \partial\Omega_1 \geq \delta > 0\}$ for some $\alpha(\delta) \to 0$ as $\delta \to 0$, and $\|\theta^\varepsilon\|_{L^\infty} \leq C$. Then,

(1) The support of the Young measure $\nu_{x,y}$ strictly excludes the stagnation point $q = 0$ and the Young measure is a Dirac mass;

(2) The sequence $(u^\varepsilon, v^\varepsilon)$ has a subsequence converging strongly in $L^2_{loc}(\Omega)$ to an entropy solution.

(iii) The boundary condition $(u, v) \cdot \mathbf{n} \geq 0$ on $\partial\Omega_1$ is satisfied in the sense of normal trace.

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Recent work: $\gamma = 3$ by G.-Q. Chen-T. Giron-S. Schulz.

Transonic flows in nozzles

Earlier works for flows in nozzles:

– Compressible flows in nozzles:

Courant-Friedrichs, Chen-Deng-Xiang, Cheng-Du-Xiang, Du-Xie-Xin, Wang-Xin, Xie-Xin, Chen-Huang-Wang-Xiang, ...

– Transonic shocks in nozzles:

S.-X. Chen, Chen-Feldman, Chen-Chen-Feldman, Chen-Yuan, Fang,-Xin, Li,-Xin-Yi, ...

– Contact discontinuity:

Bae-Park, Huang-Kuang-W.- Xiang, ...

Consider the stability of steady transonic contact discontinuity for the compressible flows in a two-dimensional (2D) finitely long nozzle:

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = 0, \\ \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = 0, \\ \partial_x\big((\rho E + p)u\big) + \partial_y\big((\rho E + p)v\big) = 0, \end{cases}$$

$$E = \frac{1}{2}(u^2 + v^2) + e(\rho, p),\ p = A(S)\rho^\gamma,\ e = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}e^{\frac{S}{c_\nu}},\ A(S) = \kappa e^{\frac{S}{c_\nu}}.$$

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$$E = \frac{1}{2}(u^2 + v^2) + e(\rho, p),\ p = A(S)\rho^\gamma,\ e = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}e^{\frac{S}{c_\nu}},\ A(S) = \kappa e^{\frac{S}{c_\nu}}.$$

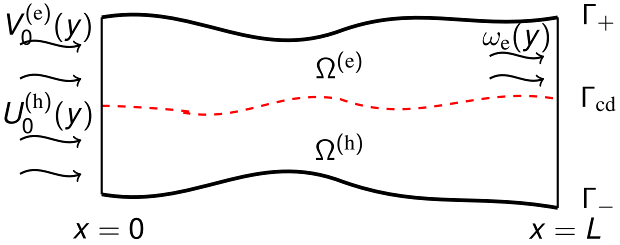
The Bernoulli function $B = \frac{1}{2}(u^2 + v^2) + \frac{\gamma p}{(\gamma-1)\rho}$ and the entropy S satisfy

$$u\partial_x B + v\partial_y B = 0 \qquad \text{and} \qquad u\partial_x S + v\partial_y S = 0.$$

Set $V = (p, B, S)$.

The domain in the nozzle:

$$\Omega := \{(x,y) \in \mathbb{R}^2 : 0 < x < L,\ g_-(x) < y < g_+(x)\},$$

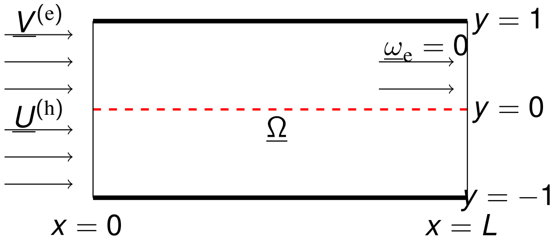


The location of the contact discontinuity:

$$\Gamma_{cd} = \{y = g_{cd}(x),\ 0 < x < L\}.$$

Background solution

Transonic flow in a flat nozzle with contact discontinuity:



The solution in the subsonic region:

$$\underline{U}^{(\mathrm{e})} := (\underline{u}^{(\mathrm{e})}, 0, \underline{p}^{(\mathrm{e})}, \underline{\rho}^{(\mathrm{e})})^\top.$$

The solution in the supersonic region:

$$\underline{U}^{(\mathrm{h})} := \left(\underline{u}^{(\mathrm{h})}, 0, \underline{p}^{(\mathrm{h})}, \underline{\rho}^{(\mathrm{h})}\right)^\top.$$

The initial incoming flow $U_0(y)$ at $x = 0$:

$$U_0(y) = \begin{cases} V_0^{(\mathrm{e})}(y), & y \in \Gamma_{\mathrm{in}}^{(\mathrm{e})}, \\ U_0^{(\mathrm{h})}(y), & y \in \Gamma_{\mathrm{in}}^{(\mathrm{h})}. \end{cases}$$

On the nozzle walls Γ_- and Γ_+:

$$\left(u^{(\mathrm{h})}, v^{(\mathrm{h})}\right) \cdot \mathbf{n}_- = 0 \text{ on } \Gamma_-, \quad \left(u^{(\mathrm{e})}, v^{(\mathrm{e})}\right) \cdot \mathbf{n}_+ = 0 \text{ on } \Gamma_+.$$

Along the contact discontinuity $y = g_{\mathrm{cd}}(x)$, the following Rankine-Hugoniot conditions hold:

$$(u, v) \cdot \mathbf{n}_{\mathrm{cd}} = 0, \quad \left[\frac{v}{u}\right] = [p] = 0, \qquad \text{on} \quad \Gamma_{\mathrm{cd}}.$$

In $\Omega^{(\mathrm{e})}$, the flow slope at the exit $\Gamma_{\mathrm{ex}}^{(\mathrm{e})}$ is given by

$$\omega^{(\mathrm{e})}(L, y) = \omega_{\mathrm{e}}(y),$$

with

$$\omega_{\mathrm{e}}(g_{\mathrm{cd}}(L)) = \frac{v}{u}(L, g_{\mathrm{cd}}(L)).$$

Stability of contact discontinuity:

– Subsonic-subsonic: *Bae-Park ('13, '19)*

– Supersonic-supersonic: *Huang-Kuang-W.- Xiang ('19)*

Problem

Huang-Kuang-W.- Xiang

For a given transonic incoming flow $U_0(y)$ at the entrance and a given flow slope $\omega_{\mathrm{e}}(y)$ at the exit $\Gamma_{\mathrm{ex}}^{(\mathrm{e})}$, find a unique piecewise smooth transonic solution $\big(U(x,y), g_{\mathrm{cd}}(x)\big)$ that is separated by the contact discontinuity Γ_{cd} satisfying the Euler system in the weak sense and the boundary conditions. The solution is a small perturbation of the background solution $(\underline{U}, 0)$.

Theorem (Main Theorem, Huang-Kuang-W.- Xiang, Ann. PDE)

There exist constants $\alpha_0 \in (0,1)$ and $\epsilon_0 > 0$ depending only on $\underline{U}$ and L, such that for any given $\alpha \in (0,\alpha_0)$ and $\epsilon \in (0,\epsilon_0)$, if

$$\left\| V_0^{(\mathrm{e})} - \underline{V}^{(\mathrm{e})} \right\|_{1,\alpha;\Gamma_{\mathrm{in}}^{(\mathrm{e})}} + \left\| U_0^{(\mathrm{h})} - \underline{U}^{(\mathrm{h})} \right\|_{1,\alpha;\Gamma_{\mathrm{in}}^{(\mathrm{h})}} + \left\| \omega_{\mathrm{e}} \right\|_{2,\alpha;\Gamma_{\mathrm{ex}}^{(\mathrm{e})}}^{(-1-\alpha,\{P_{\mathrm{e}},Q_{\mathrm{e}}\})} + \left\| g_- + 1 \right\|_{2,\alpha;\Gamma_-} + \left\| g_+ - 1 \right\|_{2,\alpha;\Gamma_+} \leq \epsilon,$$

and $\underline{M}^{(\mathrm{h})} = \frac{\underline{u}^{(\mathrm{h})}}{\underline{c}^{(\mathrm{h})}} > \sqrt{1 + \frac{1}{4}L^2}$, there exists a unique solution $(U(x,y), g_{\mathrm{cd}}) \in H^1_{\mathrm{loc}}(\Omega) \times C^{2,\alpha}([0,L])$ such that

(i) *The solution U consists of the supersonic flow $U^{(\mathrm{h})} \in C^{1,\alpha}(\Omega^{(\mathrm{h})})$ and subsonic flow $U^{(\mathrm{e})} \in C^{1,\alpha}_{(-\alpha,\Sigma^{(\mathrm{e})}\setminus\{O\})}(\Omega^{(\mathrm{e})})$ separated by $y = g_{\mathrm{cd}}(x)$, and the following estimate holds:*

$$\left\| U^{(\mathrm{e})} - \underline{U}^{(\mathrm{e})} \right\|_{1,\alpha;\Omega^{(\mathrm{e})}}^{(-\alpha,\Sigma^{(\mathrm{e})}\setminus\{O\})} + \left\| U^{(\mathrm{h})} - \underline{U}^{(\mathrm{h})} \right\|_{1,\alpha;\Omega^{(\mathrm{h})}} \leq C_0\epsilon;$$

(ii) *The contact discontinuity $y = g_{\mathrm{cd}}(x)$ is a stream line with $g_{\mathrm{cd}}(0) = 0$ and satisfies $\left\| g_{\mathrm{cd}} \right\|_{2,\alpha;\Gamma_{\mathrm{cd}}\cup\{O\}} \leq C_0\epsilon$, where $C_0 > 0$ is a constant depending only on $\underline{U}$ and L.*

Main approaches and difficulties

– Straighten the free boundary of contact discontinuity by the Euler-Lagrangian coordinate transformation, but get a new free boundary on the upper wall.

– Solve the nonlinear second-order elliptic equation in the subsonic region for the stream function.

– Solve the hyperbolic system in the supersonic region.

– Develop an iteration scheme, and show convergence by contraction.

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Many open problems!

Part 3:

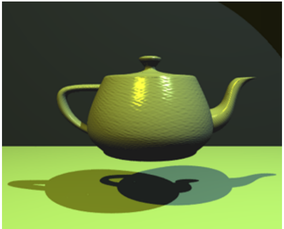
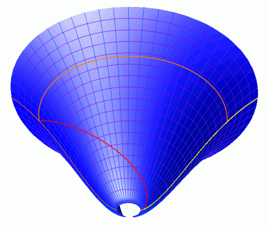
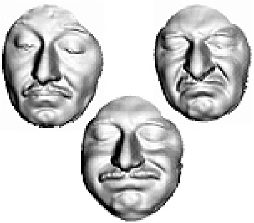
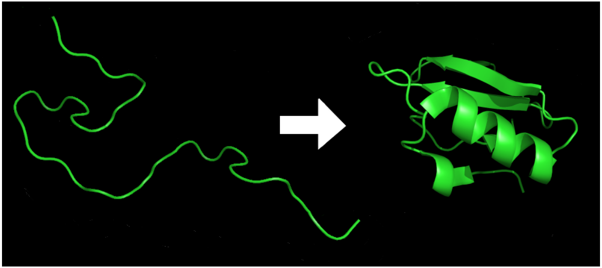
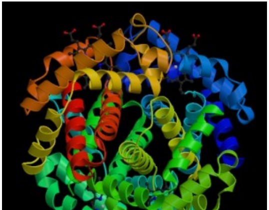
Transonic flows in isometric embeddings

Isometric Embedding

Isometric embedding in differential geometry, with applications in:

- shell theory,
- computer sciences,
- protein folding (Mathematical Challenge Ten of DARPA),
- ......

Janet, Cartan, Nash, Kuiper, Gromov, Günther, Yau, Nakamura, Nirenberg, Lin, Hong, Han, Pogorelov, Y. Li, Guan-Li, Efimov, Bryant-Griffiths-Yang, Nakamura-Maeda, Han-Khuri, Lewicka-Pakzad, Christoforou, Poole, Cao-Szekelyhidi, ...

Isometric Embedding of $\mathbb{M}^d$ into $\mathbb{R}^N$

Nash (1965), Günther (1989): smooth embeddings.

Günther (1989): *Any smooth d-dimensional compact Riemannian manifold admits a smooth (i.e. C^∞) isometric embedding in $\mathbb{R}^N$ for*

$$N = \frac{1}{2}\max\{d(d+5), d(d+3)+10\}.$$

Janet dimension:

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Example – isometric embedding of surfaces : $d = 2, \quad N = 3.$

Isometric Embedding of Surfaces in $\mathbb{R}^3$

g_{ij}, $i, j = 1, 2$: **the given metric** of a 2-D Riemannian manifold $\mathcal{M}$ defined on $\Omega \subset \mathbb{R}^2$.

The first fundamental form:

$$I := g_{11}(dx)^2 + 2g_{12}dxdy + g_{22}(dy)^2.$$

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The isometric embedding problem is to seek a map

$$\mathbf{r} : \Omega \to \mathbb{R}^3$$

such that

$$d\mathbf{r} \cdot d\mathbf{r} = I,$$

that is,

$$\partial_x \mathbf{r} \cdot \partial_x \mathbf{r} = g_{11}, \quad \partial_x \mathbf{r} \cdot \partial_y \mathbf{r} = g_{12}, \quad \partial_y \mathbf{r} \cdot \partial_y \mathbf{r} = g_{22},$$

so that $\{\partial_x \mathbf{r}, \partial_y \mathbf{r}\}$ in $\mathbb{R}^3$ are linearly independent.

The fundamental theorem of surface theory

The second fundamental form:

$$I\!I := h_{11}(dx)^2 + 2h_{12}dxdy + h_{22}(dy)^2.$$

There exists a surface in $\mathbb{R}^3$ with the fundamental forms I and $I\!I$ if (g_{ij}) and (h_{ij}) (with $(g_{ij}) > 0$) satisfy the Gauss-Codazzi system.

This theorem holds even when $(h_{ij}) \in L^\infty$ for given $(g_{ij}) \in C^{1,1}$, for which the immersion surface is $C^{1,1}$. *Mardare (2003)*

Gauss-Codazzi Equations

$$\begin{cases} \partial_x M - \partial_y L = \Gamma_{22}^{(2)} L - 2\Gamma_{12}^{(2)} M + \Gamma_{11}^{(2)} N, \\ \partial_x N - \partial_y M = -\Gamma_{22}^{(1)} L + 2\Gamma_{12}^{(1)} M - \Gamma_{11}^{(1)} N, \end{cases}$$

$$LN - M^2 = \kappa,$$ (Monge-Ampère constraint)

where

$$L = \frac{h_{11}}{\sqrt{|g|}}, \quad M = \frac{h_{12}}{\sqrt{|g|}}, \quad N = \frac{h_{22}}{\sqrt{|g|}}, \quad |g| = det(g_{ij}) = g_{11}g_{22} - g_{12}^2,$$

$$\kappa(x,y) = \frac{R_{1212}}{|g|}, \quad R_{ijkl} = g_{lm}\left(\partial_k \Gamma_{ij}^{(m)} - \partial_j \Gamma_{ik}^{(m)} + \Gamma_{ij}^{(n)}\Gamma_{nk}^{(m)} - \Gamma_{ik}^{(n)}\Gamma_{nj}^{(m)}\right),$$

$$\Gamma_{ij}^{(k)} = \frac{1}{2} g^{kl}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right).$$ (Christoffel symbol)

Mixed Type

Consider (M, N) as the state variables. If $N \neq 0$, the eigenvalues are

$$\lambda_\pm = \frac{-M \pm \sqrt{-\kappa}}{N}.$$

The Gauss-Codazzi system is

$$\text{hyperbolic if } \kappa < 0,$$
$$\text{elliptic if } \kappa > 0,$$
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The Gauss curvature κ may change sign, thus the system is of mixed hyperbolic-elliptic type.

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- Local isometric embedding of 2d and 3d manifolds with Gauss curvature changing sign cleanly:

  C.-S. Lin, Q. Han, T. Poole......

A Fluid Dynamic Formulation:

Chen-Slemrod-D.W.

$$L = \rho v^2 + p, \quad M = -\rho uv, \quad N = \rho u^2 + p,$$

$$\begin{cases} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = -(\rho v^2 + p)\Gamma_{22}^{(2)} - 2\rho uv\Gamma_{12}^{(2)} - (\rho u^2 + p)\Gamma_{11}^{(2)}, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = -(\rho v^2 + p)\Gamma_{22}^{(1)} - 2\rho uv\Gamma_{12}^{(1)} - (\rho u^2 + p)\Gamma_{11}^{(1)}, \end{cases}$$

$$\rho p q^2 + p^2 = \kappa, \qquad q^2 = u^2 + v^2.$$

Chaplygin-type gas: $p = -\frac{1}{\rho}$. The "Bernoulli" relation:

$$\rho = \frac{1}{\sqrt{q^2 + \kappa}}, \qquad p = -\sqrt{q^2 + \kappa}; \quad c^2 = q^2 + \kappa, \quad c^2 = p'(\rho) = \frac{1}{\rho^2}.$$

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L^∞ solution $\Longrightarrow$ $C^{1,1}$ immersion.

Isometric embeddings with negative Gauss curvatures

- Isometric embeddings with positive Gauss curvatures: elliptic problem, many works.
- Isometric embeddings with negative Gauss curvatures: hyperbilic problem, only a few results.

  Quote from S.-T. Yau, Review of geometry and analysis. Asian J. Math. 4 (2000), 235-278.

  “The isometric problem for surfaces of negative curvature is a very interesting nonlinear hyperbolic problem. As such, it is very difficult to prove global existence theorems for such surfaces."

L^∞ weak solutions – $C^{1,1}$ isometric immersions

1. **Fluid dynamics approach**
   (joint with G.-Q. Chen and M. Slemrod, CMP),

2. **Vanishing artificial viscosity approach**
   (joint with W. Cao and F.-M. Huang, ARMA),

3. **Finite difference approximation approach**
   (joint with W. Cao and F.-M. Huang, SIMA).

Fluid dynamics approach

Chen-Slemrod-W.

$\kappa < 0$: $\quad \kappa = -\gamma^2, \quad \gamma > 0.$

Rescale (L, M, N):

$$\tilde{L} = \frac{L}{\gamma}, \qquad \tilde{M} = \frac{M}{\gamma}, \qquad \tilde{N} = \frac{N}{\gamma}, \quad \Rightarrow \tilde{L}\tilde{N} - \tilde{M}^2 = -1.$$

A viscous approximation:

$$\begin{cases} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = R_1 + \varepsilon \partial_y^2(\rho v), \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = R_2 + \varepsilon \partial_y^2(\rho u), \end{cases}$$

where

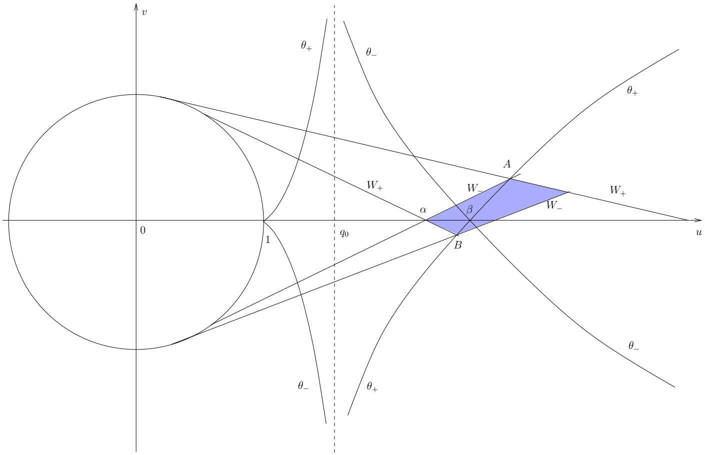
$$R_1 := -(\rho v^2 + p)\tilde{\Gamma}_{22}^{(2)} - 2\rho uv\tilde{\Gamma}_{12}^{(2)} - (\rho u^2 + p)\tilde{\Gamma}_{11}^{(2)},$$

$$R_2 := -(\rho v^2 + p)\tilde{\Gamma}_{22}^{(1)} - 2\rho uv\tilde{\Gamma}_{12}^{(1)} - (\rho u^2 + p)\tilde{\Gamma}_{11}^{(1)},$$

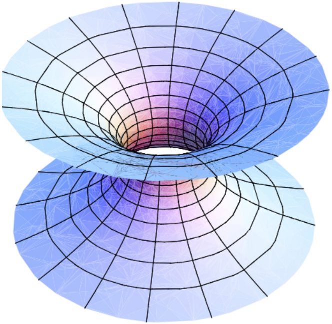
$$\tilde{\Gamma}_{11}^{(1)} = \Gamma_{11}^{(1)} + \frac{\gamma_x}{\gamma}, \qquad \tilde{\Gamma}_{12}^{(1)} = \Gamma_{12}^{(1)} + \frac{\gamma_y}{2\gamma}, \qquad \tilde{\Gamma}_{22}^{(1)} = \Gamma_{22}^{(1)},$$

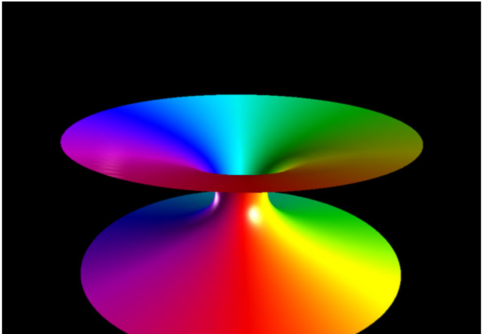
$$\tilde{\Gamma}_{11}^{(2)} = \Gamma_{11}^{(2)}, \qquad \tilde{\Gamma}_{12}^{(2)} = \Gamma_{12}^{(2)} + \frac{\gamma_x}{2\gamma}, \qquad \tilde{\Gamma}_{22}^{(2)} = \Gamma_{22}^{(2)} + \frac{\gamma_y}{\gamma}.$$

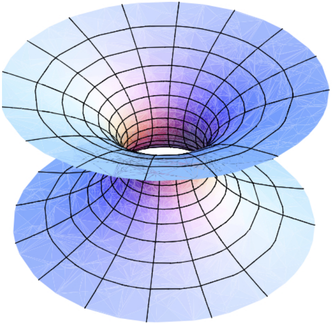
Invariant Region

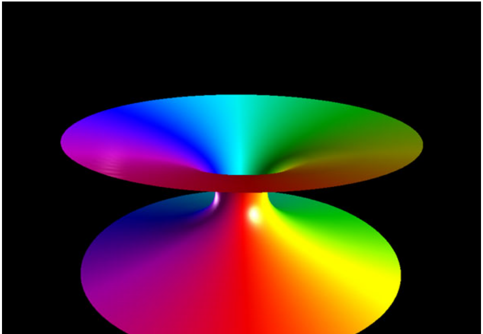


Catenoid: $g_{11} = g_{22} = (cosh(cx))^{\frac{2}{\beta^2-1}}$, $g_{12} = 0$, $\kappa(x) = -\kappa_0 E(x)^{-\beta^2}$, $c \neq 0$, $\kappa_0 > 0$, $\beta > \sqrt{2}$.









Passing the Limit

$$\begin{cases} \partial_x M - \partial_y L = \Gamma_{22}^{(2)} L - 2\Gamma_{12}^{(2)} M + \Gamma_{11}^{(2)} N, \\ \partial_x N - \partial_y M = -\Gamma_{22}^{(1)} L + 2\Gamma_{12}^{(1)} M - \Gamma_{11}^{(1)} N, \end{cases}$$

$$LN - M^2 = \kappa,$$

(Monge-Ampère constraint)

Theorem (Weak Continuity of a 2×2 Determinant)

Let $\Omega \subset \mathbb{R} \times \mathbb{R}^+ = \mathbb{R}^2_+$ be a bounded open set and $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, u_4^\varepsilon) : \Omega \to \mathbb{R}^4$ be measurable functions satisfying

$$u^\varepsilon \rightharpoonup u = (u_1, u_2, u_3, u_4) \qquad \text{in } L^2_4(\Omega),$$

and

$$\frac{\partial u_1^\varepsilon}{\partial t} + \frac{\partial u_2^\varepsilon}{\partial x}, \quad \frac{\partial u_3^\varepsilon}{\partial t} + \frac{\partial u_4^\varepsilon}{\partial x} \qquad \text{are compact in } H^{-1}_{loc}(\Omega).$$

Then there exists a subsequence (still labeled) u^ε such that

$$\begin{vmatrix} u_1^\varepsilon & u_2^\varepsilon \\ u_3^\varepsilon & u_4^\varepsilon \end{vmatrix} \rightharpoonup \begin{vmatrix} u_1 & u_2 \\ u_3 & u_4 \end{vmatrix} \qquad \text{in the sense of distributions.}$$

Vanishing artificial viscosity approach

Joint with W. Cao and F.-M. Huang.

Artificial viscosity:

$$\begin{cases} \tilde{L}_y - \tilde{M}_x = \varepsilon \tilde{L}_{xx} - \tilde{\Gamma}_{22}^2 \tilde{L} + 2\tilde{\Gamma}_{12}^2 \tilde{M} - \tilde{\Gamma}_{11}^2 \tilde{N}, \\ \tilde{M}_y - \tilde{N}_x = \varepsilon \tilde{M}_{xx} + \tilde{\Gamma}_{22}^1 \tilde{L} - 2\tilde{\Gamma}_{12}^1 \tilde{M} + \tilde{\Gamma}_{11}^1 \tilde{N}, \end{cases}$$

with

$$\tilde{L}\tilde{N} - \tilde{M}^2 = -1,$$

where

$$\tilde{L} = \frac{L}{\gamma}, \quad \tilde{M} = \frac{M}{\gamma}, \quad \tilde{N} = \frac{N}{\gamma}, \quad \kappa = -\gamma^2.$$

The eigenvalues and Riemann invariants are:

$$\frac{-\tilde{M} \pm 1}{\tilde{L}}.$$

Introduce new variables:

$$u = -\frac{\tilde{M}}{\tilde{L}}, \quad v = \frac{1}{\tilde{L}}.$$

$$\begin{cases} u_y + (uu_x - vv_x) = f(u,v) + \varepsilon u_{xx} - \frac{2\varepsilon u_x v_x}{v}, \\ v_y + (uv_x - vu_x) = g(u,v) + \varepsilon v_{xx} - \frac{2\varepsilon v_x^2}{v}, \end{cases}$$

with

$$\begin{cases} f(u,v) = -\tilde{\Gamma}^1_{22} + (\tilde{\Gamma}^2_{22} - 2\tilde{\Gamma}^1_{12})u + (2\tilde{\Gamma}^2_{12} - \tilde{\Gamma}^1_{11})u^2 + \tilde{\Gamma}^1_{11}v^2 + \tilde{\Gamma}^2_{11}(u^2 - v^2)u, \\ g(u,v) = \tilde{\Gamma}^2_{22}v + \tilde{\Gamma}^2_{12}uv + \tilde{\Gamma}^2_{11}(u^2 - v^2)v. \end{cases}$$

The eigenvalues are

$$\lambda_1 = u - v, \quad \lambda_2 = u + v.$$

The Riemann invariants are:

$$w = u + v, \quad z = u - v.$$

$$\begin{cases} w_y + \lambda_1 w_x = \varepsilon w_{xx} - \frac{2\varepsilon v_x w_x}{v} + f(u,v) + g(u,v), \\ z_y + \lambda_2 z_x = \varepsilon z_{xx} - \frac{2\varepsilon v_x z_x}{v} + f(u,v) - g(u,v). \end{cases}$$

Invariant regions

First fundamental form:

$$I = Edx^2 + 2Fdxdy + Gdy^2.$$

1. Catenoid-type surfaces:

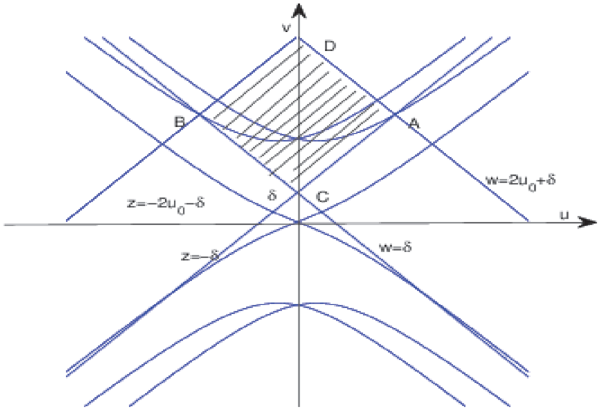
$$E(y) = (c\cosh(y/c))^{\frac{2}{\beta^2-1}}, \quad F = 0, \quad G(y) = \frac{1}{c^2(\beta^2-1)^2}E(y),$$

$$\kappa(y) = -c^2(\beta^2-1)E(y)^{-\beta^2}, \quad c \neq 0, \beta \geq \sqrt{2}.$$

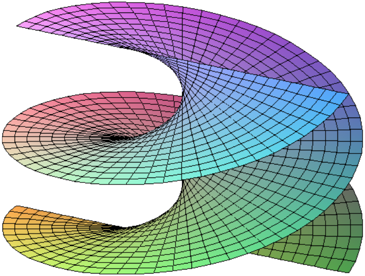
2. Helicoid-type surfaces:

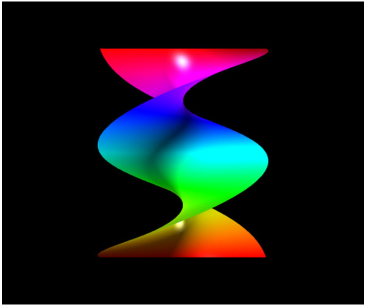
$$E(y) = c^2 + y^2, \quad F = 0, \quad G(y) = 1,$$

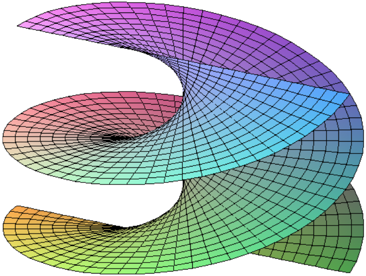
$$\kappa(y) = -\frac{c^2}{(c^2+y^2)^2}, \quad c \neq 0.$$

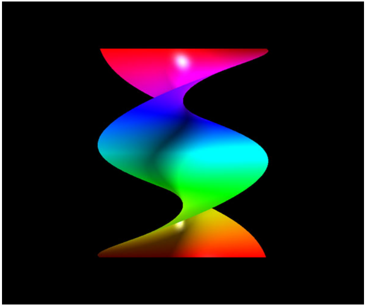


v
D
B
A
w=2u_0+δ
δ
C
z=−2u_0−δ
u
z=−δ
w=δ









Christoforou and Slemrod (2015): Gauss curvature decays as in Hong (1993).

Leading Edge
Previews

Cell

Differential Geometry Meets the Cell

Wallace F. Marshall[1,*]
[1]Department of Biochemistry and Biophysics, University of California, San Francisco, 600 16th Street, San Francisco, CA 94158, USA
*Correspondence: wallace.marshall@ucsf.edu
http://dx.doi.org/10.1016/j.cell.2013.06.032

A new study by Terasaki et al. highlights the role of physical forces in biological form by showing that connections between stacked endoplasmic reticulum cisternae have a shape well known in classical differential geometry, the helicoid, and that this shape is a predictable consequence of membrane physics.

Cell 154, July 18, 265-266, 2013.

Terasaki, M., Shemesh, T., Kasthun, N., Klemm, R.W., Schalek, R., Hayworth, K.H., Hand, A.R., Yankova, M., Huber, G., Lichtman, J.W., et al. (2013). Cell 154, July 18, 285-296.

Finite difference approximation approach: Lax-Friedrichs scheme

Joint with W. Cao and F. Huang.

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Take the metric and Gauss curvature:

$$g = B(y)^2 dx^2 + dy^2, \quad \kappa(y) = -k(y)^2,$$

and $\ln(B^2 k)$ is C^1 and nondecreasing in y.

$$\rho = \tilde{L}, \quad m = -\tilde{M},$$

$$\begin{cases} \rho_t + m_x = -\rho\dfrac{k_t}{k} - 2m\dfrac{k_x}{2k} - \dfrac{m^2 - 1}{\rho}(-BB_t), \\ m_t + (\dfrac{m^2 - 1}{\rho})_x = -2m(\dfrac{B_t}{B} + \dfrac{k_t}{k}) - \dfrac{m^2 - 1}{\rho}(\dfrac{B_x}{B} + \dfrac{k_x}{k}). \end{cases}$$

Riemann invariants:

$$w = \frac{m + 1}{\rho}, \quad z = \frac{m - 1}{\rho}.$$

$$
\begin{aligned}
w^h &= \frac{m_R^h + 1 + [-2m_R^h\left(\frac{B_t}{B} + \frac{k_t}{k}\right) - \frac{(m_R^h)^2 - 1}{\rho_R^h}\left(\frac{B_x}{B} + \frac{k_x}{k}\right)]h}{\rho_R^h + [-\rho_R^h \frac{k_t}{k} - 2m_R^h \frac{k_x}{2k} - \frac{(m_R^h)^2 - 1}{\rho_R^h}(-BB_t)]h} \\
&= \frac{w_R^h + [(w_R^h + z_R^h)\left(\frac{B_t}{B} + \frac{k_t}{k}\right) - w_R^h z_R^h\left(\frac{B_x}{B} + \frac{k_x}{k}\right)]h}{1 + [-\frac{k_t}{k} - (w_R^h + z_R^h)\frac{k_x}{2k} - w_R^h z_R^h(-BB_t)]h} \\
z^h &= \frac{m_R^h - 1 + [-2m_R^h\left(\frac{B_t}{B} + \frac{k_t}{k}\right) - \frac{(m_R^h)^2 - 1}{\rho_R^h}\left(\frac{B_x}{B} + \frac{k_x}{k}\right)]h}{\rho_R^h + [-\rho_R^h \frac{k_t}{k} - 2m_R^h \frac{k_x}{2k} - \frac{(m_R^h)^2 - 1}{\rho_R^h}(-BB_t)]h} \\
&= \frac{z_R^h + [(w_R^h + z_R^h)\left(\frac{B_t}{B} + \frac{k_t}{k}\right) - w_R^h z_R^h\left(\frac{B_x}{B} + \frac{k_x}{k}\right)]h}{1 + [-\frac{k_t}{k} - (w_R^h + z_R^h)\frac{k_x}{2k} - w_R^h z_R^h(-BB_t)]h}.
\end{aligned}
$$

$$w^h = w_R^h + hF(w_R^h, z_R^h, x, t, h)$$
$$z^h = z_R^h + hF(z_R^h, w_R^h, x, t, h),$$

where

$$F(w_R^h, z_R^h, x, t, h) = \frac{-(\frac{B_t}{B} - \frac{k_t}{2k})w_R^h - (\frac{B_t}{B} + \frac{k_t}{2k})z_R^h - BB_t(w_R^h)^2(z_R^h)}{1 - [\frac{k_t}{k} + \frac{k_x}{k}(w_R^h + z_R^h) - BB_t w_R^h z_R^h]h} + \frac{-(\frac{B_x}{B} + \frac{k_x}{2k})w_R^h z_R^h + \frac{k_x}{2k}(w_R^h)^2}{1 - [\frac{k_t}{k} + \frac{k_x}{k}(w_R^h + z_R^h) - BB_t w_R^h z_R^h]h}.$$

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& + \frac{-(\frac{B_x}{B} + \frac{k_x}{2k})w_R^h z_R^h + \frac{k_x}{2k}(w_R^h)^2}{1 - [\frac{k_t}{k} + \frac{k_x}{k}(w_R^h + z_R^h) - BB_t w_R^h z_R^h]h}.
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L^∞ weak solution: Uniform estimate, convergence, and consistency.

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\end{aligned}
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L^∞ weak solution: Uniform estimate, convergence, and consistency.

Recent work: S. Li (weak solution for more metrics), ...

Smooth isometric immersion

Cao-Han-Huang-W. (2023)

Let $(\mathcal{M}, g)$ be a smooth complete simply connected surface with a negative Gauss curvature K and

$$\int_{\mathcal{M}} |K| dA_g < \infty,$$

where dA_g is the area element of g. Assume that in some geodesic polar coordinates (θ, ρ) on $(\mathcal{M}, g)$, K has the decomposition

$$\rho^{2+\gamma}|K|(\theta, \rho) = \overline{K}(\rho) a^2(\theta, \rho) \quad \text{for } \rho \text{ large},$$

where $\gamma \in (0, 1)$ is a constant and $\overline{K}$ and a are positive functions such that $\overline{K}(\rho)$ is monotone for ρ large, and $a, a^{-1}, \partial_\theta^i \log a, \rho\partial_\theta^i \partial_\rho \log a$ are bound for $i = 1, 2, 3$,

$$\int_1^\infty \max_\theta |\partial_\rho a| d\rho < \infty.$$

Then, $(\mathcal{M}, g)$ admits a smooth isometric immersion in $\mathbb{R}^3$.

Isometric Embedding of $\mathbb{M}^d$ into $\mathbb{R}^N$, $d \geq 3$

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Not elliptic: S.-S. Chern and H. Levy.

Gauss-Codazzi-Ricci System

The Gauss equations:

$$h^a_{ji}h^a_{kl} - h^a_{ki}h^a_{jl} = R_{ijkl};$$

The Codazzi equations:

$$\frac{\partial h^a_{lj}}{\partial x^k} - \frac{\partial h^a_{kj}}{\partial x^l} + \Gamma^m_{lj}h^a_{km} - \Gamma^m_{kj}h^a_{lm} + \kappa^a_{kb}h^b_{lj} - \kappa^a_{lb}h^b_{kj} = 0;$$

The Ricci equations:

$$\frac{\partial \kappa^a_{lb}}{\partial x^k} - \frac{\partial \kappa^a_{kb}}{\partial x^l} - g^{mn}\left(h^a_{ml}h^b_{kn} - h^a_{mk}h^b_{ln}\right) + \kappa^a_{kc}\kappa^c_{lb} - \kappa^a_{lc}\kappa^c_{kb} = 0.$$

R_{ijkl}: Riemann curvature tensor,
h^a_{ij}: Coefficients of the second fundamental form,
κ^a_{lb}: Coefficients of the connection form on the normal bundle,
$1 \leq a, b \leq N - d$; $1 \leq i, j, k, l, m, n \leq d$.

The Div-Curl Structure

For $w = (w_1, w_2, \cdots, w_d)$, curl $w := (\partial_j w_i - \partial_i w_j)_{1 \le i,j \le d}$.

Codazzi equations: $\quad k < l$,

$$\mathrm{div}(\underbrace{\overbrace{0, \cdots, h^a_{lj}}^{k}, 0, \cdots, -h^a_{kj}}_{l}, 0, \cdots, 0) + l.o.t = 0,$$

$$\mathrm{curl}(h^a_{1j}, h^a_{2j}, \cdots, h^a_{dj}) + l.o.t = 0,$$

Ricci equations:

$$\mathrm{div}(\underbrace{\overbrace{0, \cdots, 0, \kappa^a_{lb}}^{k}, 0, \cdots, -\kappa^a_{kb}}_{l}, 0, \cdots, 0) + l.o.t = 0,$$

$$\mathrm{curl}(\kappa^a_{1b}, \kappa^a_{2b}, \cdots, \kappa^a_{db}) + l.o.t = 0.$$

Scalar products yield the quadratic terms.

Div-Curl Lemma

Let $\Omega \subset \mathbb{R}^d, d \geq 2$, be open bounded. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that, for any $\varepsilon > 0$, two fields $u^\varepsilon \in L^p(\Omega; \mathbb{R}^d)$ and $v^\varepsilon \in L^q(\Omega; \mathbb{R}^d)$ satisfy the following:

i. $u^\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega; \mathbb{R}^d)$ as $\varepsilon \to 0$;
ii. $v^\varepsilon \rightharpoonup v$ weakly in $L^q(\Omega; \mathbb{R}^d)$ as $\varepsilon \to 0$;
iii. $\operatorname{div} u^\varepsilon$ are confined in a compact subset of $W^{-1,p}_{loc}(\Omega; \mathbb{R})$;
iv. $\operatorname{curl} v^\varepsilon$ are confined in a compact subset of $W^{-1,q}_{loc}(\Omega; \mathbb{R}^{d\times d})$.

Then the scalar product of u^ε and v^ε are weakly continuous:

$$u^\varepsilon \cdot v^\varepsilon \longrightarrow u \cdot v$$

in the sense of distributions.

Weak Continuity:

Chen-Slemrod-D.W. (PAMS)

The weak limit of a sequence of solutions to the Gauss-Codazzi-Ricci system is still a sloution.

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Let $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ be a sequence of solutions to the Gauss-Codazzi-Ricci system, which is uniformly bounded in L^p, $p > 2$. Then the weak limit vector field $(h_{ij}^{a}, \kappa_{lb}^{a})$ of the sequence $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ in L^p is still a solution to the Gauss-Codazzi-Ricci system.

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3-D manifold into $\mathbb{R}^6$: local isometric embedding,

Bryant-Griffiths-Yang (83), Chen-Clelland-Slemrod-W. -Yang (18).

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More recent works: G.-Q. Chen- S. Li, Chen-Giron, ...

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Thank You!