Diffusion Limited Aggregation

Introduction

Diffusion Limited Aggregation (DLA) is one of the most puzzling models in probability and statistical physics. There are many thousands of physics papers discussing this model and only one mathematical theorem.

Originally it was proposed by Sander and Witten [5] as a model for irreversible colloidal aggregation. It was soon realized that it can be used to model many phenomena where diffusion is the main mean of transport: bacteria growth, mineral deposition (see figure) and many others. In this model one starts with a single particle at the origin and then sends particles from infinity one by one. Particles perform random walk on the lattice (or Brownian motion in the off-lattice case) until they hit the aggregation. When the particle hits the aggregation, it sticks to it and the next particle is sent from infinity.

It is not immediately obvious that "sending particles from infinity" makes sense, but in fact it does. Moreover, there are several ways to give a rigorous definition and they all are equivalent. In the off lattice case where the particles are discs that perform the Brownian motion, the simplest way it start from a point which is chosen uniformly on a circle which encloses the aggregation. In the lattice case the situation is a bit more complicated: the disc should be large enough, so that its lattice approximation is sufficiently good.

In both cases, after choosing the starting point, one has to simulate Brownian trajectory or random walk until it hits the the existing aggregation. The straight-forward implementation of this is very time consuming, but there are more effective algorithms that allow to make large jumps when the particle is far away from the aggregation.

Manganese deposition on a limestone

There are many questions about long time behaviour of DLA: What is the growth rate? Is it a regular fractal? What is the Hausdorff or Minkowski dimensions? Are there deep fjords? Is here a non-trivial scaling limit? Is there a difference between lattice and off-lattice models?

These and many other questions were addressed by many physics papers and there were many computer simulations as well as real experiments. The only mathematical theorem about DLA is due to Kesten who proved that the diameter of the aggregation after N steps is $O(N^{2/3})$.

Project

This project can be developed in several direction. They all involve simulation of at least one version of DLA and analysis of the data.

Task 1: Background. There are very few books that discuss DLA and its properties. I suggest to start with the DLA section in [2]. For one of algorithms to simulate off lattice DLA you can look at [4]. If you would prefer to study DLA on the square lattice (or any other regular lattice) you should read about random walks and two-dimensional discrete potential theory [3, Section 6.6]. It would be beneficial to survey other original research papers about DLA in order to understand the current state of the art.

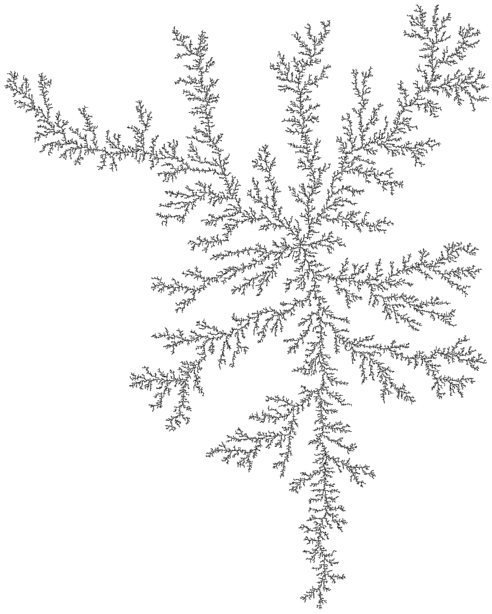

Figure 1: A DLA cluster of size 2000 on the square lattice

Task 2: Simulations. After preliminary reading you should write a code to simulate DLA clusters. The main choice at this stage is which DLA model do you want to study: lattice or off-lattice. Actually there is the third possibility: simulate both and study to what extent they are similar. Both versions are interesting, but they present different computational difficulties, some things are easier on the lattice, some are easier (or faster) off-lattice. Most of lattice simulations are done on the square lattice, but it is also possible to consider a triangular lattice or some other periodic tiling of the plane or even more complicated graphs such as random tilings or a Penrose tiling.

You should try to code this as effectively as possible. How fast is your code will determine how large are the clusters that you can sample. This will determine how accurate are computations based on these samples.

Task 3: Analysis. With sufficiently many samples you can compute various statistics that describe the growth rate and the local structure of DLA clusters. The simplest statistics is the growth rate. It is believed that the aggregate of n particles has diameter of order n^d where $1/2 \leq d \leq 1$. You can use the samples to estimate the growth rate d. Alternatively, one can compute a closely related quantity: Minkowski dimension of the cluster.

The cluster will have a tree-like structure. One can study the structure of branches which are large, but much smaller than the entire cluster. It will be interesting to compare branches that are deep inside the cluster (old branches) and branches that are close to the external perimeter (new branches). Another possibility is to compare lattice and off-lattice branches.

Task 4: Analysis (optional). Since the growth of DLA is governed by the exit distribution of the random walk (or Brownian motion) it is important to understand this distribution. One way of doing this is through the so called multifractal formalism. Roughly speaking it is about counting number of sites with given probability that a new purticle will be attached there.

References

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