Groups

ASO course Trinity 2024

Example sheet 1

1. Let $K \trianglelefteq G$ and let $\bar{H} \leq G/K$. Let $\pi : G \to G/K$ denote the quotient map $g \mapsto gK$. Show that

$$H = \pi^{-1}(\bar{H}) = \{g \in G : gK \in \bar{H}\}$$

is a subgroup of G, containing K as a normal subgroup, with $H/K = \bar{H}$. Show further that if $\bar{H} \trianglelefteq G/K$ then $H \trianglelefteq G$.

2. The dihedral group D_{2n} has presentation

$$\langle a, b \mid a^n = b^2 = 1,\ bab^{-1} = a^{-1} \rangle$$

Verify that this group has $2n$ elements, all of the form a^i or ba^i, and that $(ba^i)^2 = 1$. Interpret this geometrically.

3. Identify the following groups from their presentation

(i) $G_1 = \langle x \mid x^6 = 1 \rangle$,
(ii) $G_2 = \langle x, y \mid xy = yx \rangle$,
(iii) $G_3 = \langle x, y \mid x^3y = y^2x^2 = x^2y \rangle$,
(iv) $G_4 = \langle x, y \mid xy = yx, x^5 = y^3 \rangle$,
(v) $G_5 = \langle x, y \mid xy = yx, x^4 = y^2 \rangle$.

[For G_4 you may wish to consider the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ given by $(a, b) \mapsto 3a + 5b$].

4. Let $G = \langle x, y \mid x^2 = y^2 = 1 \rangle$.

(i) Let $z = xy$. Show that every element of G can be written as z^k or yz^k where k is an integer.

(ii) Deduce that G is isomorphic to the *infinite dihedral group* D_∞, namely the isometry group of the integers $\mathbb{Z}$, considered as a subset of the real line with the Euclidean metric.

(iii) Show that

$$G = \langle y, z | y^2 = 1, yzy^{-1} = z^{-1} \rangle.$$

is another presentation of the group.

5. Let G be a non-Abelian group of order 8.

(i) Show that G has an element a of order 4.

(ii) Let $A = \langle a \rangle$ and let $b \in G - A$. Show that $bab^{-1} = a^{-1}$ and either $b^2 = 1$ or $b^2 = a^2$.

(iii) Deduce that there are, up to isomorphism, exactly two non-Abelian groups of order 8, and five groups of order 8 in total.

Show that one of the non-Abelian groups may be identified with the *quaternion group*

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\},$$

where we have the usual quaternionic relations

$$i^2 = j^2 = k^2 = -1 \quad : \quad ij = k = -ji.$$

6. Write down all possible composition series of the following groups and verify the Jordan-Hölder Theorem for them:

$$C_{18},\ \ D_{10},\ \ D_8,\ \ Q_8.$$

7. Let H and K be subgroups of a group G. Show that

$$HK = \{hk : h \in H,\ k \in K\}$$

is a subgroup of G if and only if $HK = KH$.

8. Show that $(\mathbb{Q}, +)$ is not finitely generated.

9. Let G be a group all of whose non-identity elements have order 2. Show that G is abelian. Give an example of a non-abelian group, all of whose non-identity elements have order 3.