Mathematical Modelling and Scientific Computing

Numerical Solution of Differential Equations

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1. Let $\Omega = (0,1)$ and $b, f \in C(\bar{\Omega})$ be given functions and $u_L, u_R \in \mathbb{R}$ be given nonnegative constants. Consider the elliptic partial differential equation

$$-u''(x) + b(x)u'(x) + u(x) = f(x), \qquad x \in \Omega, \tag{1a}$$

$$u(0) = u_L, \tag{1b}$$

$$u(1) = u_R, \tag{1c}$$

(a) [3 marks] On the uniform finite difference mesh

$$\bar{\Omega}_h := \{x_i := ih,\ i = 0, \dots, N\}$$

of spacing $h := 1/N$, where $N \geqslant 2$, formulate a finite difference approximation $\{U_i : 0 \leqslant i \leqslant N\}$ to (1) of the form

$$\mathcal{L}_h U_i = f_i, \qquad 1 \leqslant i \leqslant N-1,$$

using the three-point stencil for the second-order term $-u''$ and the two-point central difference operator for the first order term u'.

(b) [6 marks] Show that if $f < 0$ on $\bar{\Omega}$ and $\|b\|_{C(\bar{\Omega})} h \leqslant 2$, then U satisfies

$$\max_{0 \leqslant i \leqslant N} U_i = \max\{u_L, u_R\}.$$

(c) [6 marks] Suppose that there exists $\delta > 0$ such that $\|b\|_{C(\bar{\Omega})} h \leqslant 2 - \delta$. Show that there exists $\lambda > 0$ such that the mesh function $W_i := e^{\lambda x_i}$ satisfies

$$\mathcal{L}_h W_i < 0 \qquad 1 \leqslant i \leqslant N-1.$$

Then, under the same assumptions, show that if $f \leqslant 0$, then U satisfies

$$\max_{0 \leqslant i \leqslant N} U_i = \max\{u_L, u_R\}.$$

(d) [4 marks] Suppose that $u_L = u_R = 0$. Show that if $\|b\|_{C(\bar{\Omega})} h \leqslant 2$, then U satisfies

$$\max_{1 \leqslant i \leqslant N-1} |U_i| \leqslant \max_{1 \leqslant i \leqslant N-1} |f(x_i)|.$$

[*Hint: Do not use parts (b) or (c).*]

(e) [6 marks] Define the consistency error φ_i of your scheme in (a) at the mesh-point x_i, $i = 1, 2, \dots, N-1$. Assuming that $u \in C^4(\bar{\Omega})$, show that

$$\max_{1 \leqslant i \leqslant N-1} |\varphi_i| \leqslant Ch^2 \left(\|b\|_{C(\bar{\Omega})} \|u'''\|_{C(\bar{\Omega})} + \|u''''\|_{C(\bar{\Omega})} \right),$$

where C is a positive constant that you should specify. Conclude that if $\|b\|_{C(\bar{\Omega})} h \leqslant 2$, then

$$\max_{0 \leqslant i \leqslant N} |u(x_i) - U_i| \leqslant Ch^2 \left(\|b\|_{C(\bar{\Omega})} \|u'''\|_{C(\bar{\Omega})} + \|u''''\|_{C(\bar{\Omega})} \right).$$

2. Let $\Omega := (0,1)^2$, $b \in \mathbb{R}$ be a given constant, and $c, f \in C(\bar{\Omega})$ be given functions. Consider the elliptic partial differential equation

$$-\Delta u + b\frac{\partial u}{\partial y} + c(x,y)u = f, \qquad \text{in } \Omega, \tag{2a}$$

$$u = 0, \qquad \text{on } \partial\Omega. \tag{2b}$$

(a) [7 marks] Suppose that $u \in C^2(\bar{\Omega})$. Show that

$$\int_\Omega |\nabla u(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \{f(x,y)u(x,y) - c(x,y)u^2(x,y)\}\,\mathrm{d}x\,\mathrm{d}y.$$

[*Hint: The identity* $\frac{\partial(u^2)}{\partial y} = 2u\frac{\partial u}{\partial y}$ *may be helpful.*]
Then, find a positive constant $M_0 > 0$ such that if

$$\|c\|_{C(\bar{\Omega})} \leqslant M_0, \tag{3}$$

then any solution $u \in C^2(\bar{\Omega})$ to the partial differential equation (2) satisfies

$$\|u\|_{H^1(\Omega)} \leqslant C_0\|f\|_{L^2(\Omega)},$$

where C_0 is a constant you should specify. Conclude that if (3) holds, then $C^2(\bar{\Omega})$ solutions to (2) are unique.
[*You may use the Poincaré-Friedrichs inequality without proof.*]

(b) [3 marks] On the uniform finite difference mesh

$$\bar{\Omega}_h := \{(x_i, y_j) : x_i := ih,\ y_j := jh,\ i,j = 0,\dots,N\}$$

of spacing $h := 1/N$ in both coordinate directions, where $N \geqslant 2$, formulate a finite difference approximation to (2) using the five-point stencil for the second-order term $-\Delta u$ and the two-point central difference operator for the first-order term $\frac{\partial u}{\partial y}$.

(c) [7 marks] Find a positive constant $M_1 > 0$ independent of h such that if

$$\|c\|_{C(\bar{\Omega})} \leqslant M_1, \tag{4}$$

then any solution U to the finite difference scheme in (b) satisfies

$$\|U\|_{1,h} \leqslant C_1\|f\|_h,$$

where C_1 is a constant you should specify and $\|\cdot\|_{1,h}$ is a discrete H^1 norm that you should specify.
Conclude that your finite difference scheme has a solution and that the solution is unique.
[*You may use the discrete Poincaré-Friedrichs inequality without proof.*]

(d) [8 marks] Define the consistency error $\varphi_{i,j}$ of your scheme in (b) at the mesh-point (x_i, y_j), $i,j = 1,2,\dots,N-1$. Assuming that $u \in C^4(\bar{\Omega})$, show that

$$\max_{1\leqslant i,j\leqslant N-1} |\varphi_{i,j}| \leqslant C_2 h^2 \left(|b|\left\|\frac{\partial^3 u}{\partial y^3}\right\|_{C(\bar{\Omega})} + \left\|\frac{\partial^4 u}{\partial x^4}\right\|_{C(\bar{\Omega})} + \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{C(\bar{\Omega})}\right),$$

where C_2 is a positive constant that you should specify. Then, show that there exists a positive constant C_3, that you should specify in terms of C_1 and C_2, such that if (4) holds, then

$$\|u - U\|_{1,h} \leqslant C_3 h^2 \left(|b|\left\|\frac{\partial^3 u}{\partial y^3}\right\|_{C(\bar{\Omega})} + \left\|\frac{\partial^4 u}{\partial x^4}\right\|_{C(\bar{\Omega})} + \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{C(\bar{\Omega})}\right).$$

3. Consider the initial value problem

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + u = \kappa\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,\ 0 < t \leqslant T, \tag{5a}$$

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty, \tag{5b}$$

where a, κ, and T are strictly positive real numbers, and u_0 is a real-valued, bounded, and continuous function of $x \in (-\infty, \infty)$.

(a) [5 marks] Suppose that $\theta \in [0,1]$. Formulate the θ-scheme, with $\theta = 1$ corresponding to the backward Euler scheme, for the numerical solution of (5) on a mesh with uniform spacings $\Delta x = 1/N$ and $\Delta t = T/M$ in the x and t coordinate directions, respectively, where $N \geqslant 2$ and $M \geqslant 1$ are integers. Use the two-point backward difference operator for the first order spatial derivative and denote the solution by U_j^m.

(b) [10 marks] Suppose that

$$\|U^0\|_{\ell^2} := \left(\Delta x \sum_{j=-\infty}^{\infty} |U_j^0|^2\right)^{1/2}$$

is finite. Find a complex valued function λ such that

$$\hat{U}^m(k) = [\lambda(k)]^m \hat{U}^0(k), \qquad k \in [-\pi/\Delta x, \pi/\Delta x],$$

for all $m = 0, 1, \ldots, M$, where $\hat{U}^m$ is the semi-discrete Fourier transform of $\{U_j^m\}$:

$$\hat{U}^m(k) := \Delta x \sum_{j=-\infty}^{\infty} U_j^m e^{-ikx_j}, \qquad k \in [-\pi/\Delta x, \pi/\Delta x].$$

Then, show that the backward Euler scheme ($\theta = 1$) satisfies

$$\|U^m\|_{\ell^2} \leqslant \left(\frac{1}{1+\Delta t}\right)^m \|U^0\|_{\ell^2}, \qquad 1 \leqslant m \leqslant M,$$

for any choice of Δx and Δt.

[*You may use the discrete version of Parseval's identity for the semidiscrete Fourier transform without proof.*]

(c) [10 marks] Suppose that u is smooth in space and time. Define the consistency error T_j^m for the θ-scheme in (a) and show that that the backward Euler scheme ($\theta = 1$) has consistency error

$$T_j^m = \mathcal{O}(\Delta t + \Delta x), \qquad j \in \mathbb{Z},\ m = 0, 1, \ldots, M-1.$$

Modify the finite difference scheme in (a) so that the Crank-Nicolson scheme ($\theta = 1/2$) has consistency error

$$T_j^m = \mathcal{O}((\Delta t)^2 + (\Delta x)^2), \qquad j \in \mathbb{Z},\ m = 0, 1, \ldots, M-1.$$

Prove that your modification has the above consistency error.

4. Consider the advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad -\infty < x < \infty,\ 0 < t \leqslant T, \tag{6a}$$

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty, \tag{6b}$$

where a and T be a positive constants and u_0 is a real-valued, bounded, continuous function of $x \in (-\infty, \infty)$.

We discretize space-time $(-\infty,\infty) \times [0,T]$ with uniform spacings $\Delta x = 1/N$ and $\Delta t = T/M$ in the x and t coordinate directions, respectively, where $N \geqslant 2$ and $M \geqslant 1$ are integers. The so-called *Beam-Warming scheme* for (6) is

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + \frac{a}{2\Delta x}\left(3U_j^m - 4U_{j-1}^m + U_{j-2}^m\right) = \frac{a^2\Delta t}{2(\Delta x)^2}\left(U_j^m - 2U_{j-1}^m + U_{j-2}^m\right), \tag{7a}$$

$$U_j^0 = u_0(j\Delta x), \tag{7b}$$

where $j \in \mathbb{Z}$ and $m = 0, 1, \ldots, M-1$.

(a) [10 marks] Show that

$$u_j^{m+1} - \frac{(a\Delta t)^2}{2(\Delta x)^2}\left(u_j^m - 2u_{j-1}^m + u_{j-2}^m\right) = \left[u - a(\Delta t)\frac{\partial u}{\partial x}\right]_j^m + \mathcal{O}((\Delta t)^3 + (\Delta t)^2(\Delta x)), \tag{8}$$

where $u_j^m := u(j\Delta x, m\Delta t)$, $j \in \mathbb{Z}$, and $0 \leqslant m \leqslant M-1$. You may assume that u has as many bounded derivatives as necessary for your arguments.
[*Hint: You may want to relate* $\frac{\partial^2 u}{\partial t^2}$ *to* $\frac{\partial^2 u}{\partial x^2}$.]

(b) [5 marks] Define the consistency error T_j^m, $j \in \mathbb{Z}$, $0 \leqslant m \leqslant M-1$, for the scheme (7) and show that it satisfies

$$T_j^m = \mathcal{O}((\Delta t)^2 + (\Delta x)^2 + (\Delta t)(\Delta x)), \qquad j \in \mathbb{Z},\ 0 \leqslant m \leqslant M-1. \tag{9}$$

You may assume that u has as many bounded derivatives as necessary for your arguments and that (8) holds regardless of your answer for part (a).

(c) [10 marks] Find a complex valued function λ of the form

$$\lambda(k) = \alpha + \beta e^{-\mathrm{i}k\Delta x} + \gamma e^{-2\mathrm{i}k\Delta x},$$

such that

$$\hat{U}^m(k) = [\lambda(k)]^m \hat{U}^0(k), \qquad k \in [-\pi/\Delta x, \pi/\Delta x],$$

for all $m = 0, 1, \ldots, M$, where α, β, and γ are constants that you should specify. Here, $\hat{U}^m$ is the semi-discrete Fourier transform of $\{U_j^m\}$:

$$\hat{U}^m(k) := \Delta x \sum_{j=-\infty}^{\infty} U_j^m e^{-\mathrm{i}kx_j}, \qquad k \in [-\pi/\Delta x, \pi/\Delta x].$$

Show that

$$|\lambda(k)|^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\beta(\alpha+\gamma)\cos(k\Delta x) + 2\alpha\gamma\cos(2k\Delta x),$$

and

$$\frac{d}{dk}|\lambda(k)|^2 = 2(\Delta x)\mu(\mu-2)(\mu-1)^2\sin(k\Delta x)(1-\cos(k\Delta x)), \quad \text{where } \mu = \frac{a\Delta t}{\Delta x}.$$

Conclude that the Beam-Warming scheme is practically stable if $0 \leqslant \mu \leqslant A$, where A is a positive constant you should specify.
[*You may use without proof the result that if* $|\lambda(k)| \leqslant 1$ *for* $k \in [-\pi/\Delta x, \pi/\Delta x]$, *then the scheme is practically stable.*]