C2.6 Introduction to Schemes Sheet 4

Hilary 2024

(1) (B) Let $f : X \to Y$ be a morphism of schemes and let $y \in Y$. Prove that the schematic fiber $X_y := X \times_Y y$ is homeomorphic to the topological fiber $f^{-1}(y)$.

(2) (B) Let X be a separated scheme. Show that, for any affine opens $U_1, \dots, U_m \subseteq X$, $U_1 \cap \cdots \cap U_m$ is affine.

(3) (B) Consider the quasicoherent sheaf $\mathcal{F} := \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(n)$ on $\mathbb{P}^r_k$. Let $S := k[x_0, \dots, x_r]$ and let $U_i := \{x_i \neq 0\} \subseteq \mathbb{P}^r_k$ and $U_{i_0\dots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$.

Prove (without referring to the Proj construction), that $\mathcal{F}(U_{i_0\dots i_p}) = S_{x_{i_0}\cdots x_{i_p}}$ (the localization of S at the element $x_{i_0}\cdots x_{i_p}$), and that this is an isomorphism of graded rings, where S has the natural grading by $\deg(x^{\ell_1\dots\ell_m}_{d_1\dots d_m}) := \ell_1 + \cdots + \ell_m$.

(4) (B) Prove that $H^1(\mathbb{P}^r_k, \mathcal{O}(d)) = 0$ for $0 < i < r$. Use induction on r.

For $r > 1$, use the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r_k}(-1) \xrightarrow{\cdot x_r} \mathcal{O}_{\mathbb{P}^r_k} \to i_*\mathcal{O}_H \to 0,$$

where $H := Z(x_r)$ and $i : H \hookrightarrow \mathbb{P}^r_k$ is the inclusion. (Note that his sequence is exact after tensoring over $\mathcal{O}_{\mathbb{P}^r_k}$ with the line bundle $\mathcal{O}(n)$, then use the long exact sequence on cohomology and the induction hypothesis).

(5) (B) Let X be an integral Noetherian separated scheme, regular in codimension 1, and let f be a nonzero rational function on X. Prove that $\mathrm{div}(f)$ is in fact a Weil divisor, i.e., that the sum in the definition of $\mathrm{div}(f)$ is finite, not infinite.

(6) (B) Prove the "excision sequence" for the Weil class group. Let X be an integral Noetherian separated scheme, regular in codimension 1. Show that if $Z \subset X$ is an integral closed subscheme, with $\operatorname{codim} Z = 1$, then the sequence

$$\mathbb{Z} \xrightarrow{1\mapsto[Z]} \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$$

is exact. Deduce that if $U := \mathbb{P}^n_k \setminus$ (a degree d hypersurface), then $\mathrm{Cl}(U) \simeq \mathbb{Z}/d\mathbb{Z}$.

(7)(B) Let $X := Z(f) \subseteq \mathbb{P}^2_k$, where f is a degree d homogeneous equation such that $[1 : 0 : 0] \in \mathbb{P}^2_k \setminus X$; here $[x_0, x_1, x_2]$ are homogeneous coordinates on $\mathbb{P}^2_k$. Let $U_1 := X \cap \{x_1 \neq 0\}$ and $U_2 := X \cap \{x_2 \neq 0\}$.

a) Check that U_1 and U_2 are affine opens of X, and that X is separated.
b) Use the cover $\{U_1, U_2\}$ of X to compute that:
   - $\dim H^0(X, \mathcal{O}_X) = 1$,
   - $\dim H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}$.