C2.6 Introduction to Schemes Sheet 4

Hilary 2024

(1) (B) Let $f : X \to Y$ be a morphism of schemes and let $y \in Y$. Prove that the underlying topological space of the schematic fiber $X_y := X \times_Y \operatorname{Spec} \kappa(y)$ is homeomorphic to the topological fiber $f^{-1}(y) \subseteq |X|$.

Solution. We claim that the homeomorphism is induced by the first projection $p : X \times_Y \operatorname{Spec} \kappa(y) \to X$.

If $V = \operatorname{Spec} A \subseteq Y$ is any affine open neighbourhood of y, so that $y : \operatorname{Spec} \kappa(y) \to X$ factors through V, then by properties of fiber product one has $X_y = (X \times_Y V) \times_V \operatorname{Spec} \kappa(y) = f^{-1}(V) \times_V \operatorname{Spec} \kappa(y)$. Hence, we reduce to the case when Y is affine.

If $U \subseteq X$ is any open subset, then $p^{-1}(U) = U \times_Y \operatorname{Spec} \kappa(y)$. Hence, we may assume that $X = \operatorname{Spec} B$ is also affine.

Say that $y = \mathfrak{p} \in \operatorname{Spec}(A)$ and f corresponds to the ring map $\varphi : A \to B$. At the level of rings, the morphism induced by p is the composite

$$B \to B \otimes_A A_{\mathfrak{p}} \to B \otimes_A \kappa(\mathfrak{p});$$

the first map is a localization and the second is the quotient. Hence, by Sheet 1, we see that p induces a homeomorphism from X_y and the set of $\mathfrak{q} \in X = \operatorname{Spec} B$ such that $\mathfrak{q} \supseteq \mathfrak{p}B$ and $\mathfrak{q} \cap \varphi(A \setminus \mathfrak{p}) = \emptyset$. This is equivalent to $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, proving the claim.

(2) (B) Let X be a separated scheme. Show that, for any affine opens $U_1, \ldots, U_m \subseteq X$, $U_1 \cap \cdots \cap U_m$ is affine.

Solution. By induction, it is enough to treat the case $m = 2$. Say $U_i = \operatorname{Spec} A_i$. We will prove that the canonical morphism

$$U_1 \cap U_2 = U_1 \times_X U_2 \to U_1 \times_{\operatorname{Spec} \mathbb{Z}} U_2 = \operatorname{Spec} A_1 \otimes_{\mathbb{Z}} A_2$$

is a closed immersion. Indeed, this map is canonically identified with

$$\operatorname{id} \times \Delta_X \times \operatorname{id} : U_1 \times_X X \times_X U_2 \to U_1 \times_X (X \times_{\operatorname{Spec} \mathbb{Z}} X) \times_X U_2,$$

which is a closed immersion since X is separated (and closed immersions are stable under base change).

(3) (B) Consider the quasicoherent sheaf $\mathcal{F} := \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(n)$ on $\mathbb{P}^r_k$. Let $S := k[x_0, \ldots, x_r]$ and let $U_i := \{x_i \neq 0\} \subseteq \mathbb{P}^r_k$ and $U_{i_0 \ldots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$.

Prove (without referring to the Proj construction), that $\mathcal{F}(U_{i_0 \ldots i_p}) = S_{x_{i_0} \cdots x_{i_p}}$ (the localization of S at the element $x_{i_0} \cdots x_{i_p}$), and that this is an isomorphism of graded rings, where S has the natural grading by $\deg(x_{d_1 \ldots d_m}^{\ell_1 \ldots \ell_m}) := \ell_1 + \cdots + \ell_m$.

Solution. First, let us recall the construction of the line bundles $\mathcal{O}(d)$. Recall that $\mathbb{P}^r_k = \bigcup_{i=0}^r U_i$ where $U_i = \operatorname{Spec} k\left[\frac{x_0}{x_i}\ldots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_r}{x_i}\right]$. Let $K := k(x_0,\ldots,x_r)_0$. Then we define $\mathcal{O}(d)(U_i) := \mathcal{O}_{\mathbb{P}^r_k}(U_i)\cdot x_i^d \subseteq K$ together with transitions from U_i to U_j given by mutliplication by $\alpha_{ij} := (x_i/x_j)^d$. In other words one has $\mathcal{O}(d)(U_i) = (S_{x_i})_d$, where the $(\cdot)_d$ denotes the degree d homogeneous elements.

Now it follows immediately by the "tilde" construction, and properties of localization, that one has $\mathcal{O}(d)(U_{i_0\ldots i_p}) = (S_{x_{i_0}\cdots x_{i_p}})_d$.

If $i : U_{i_0,\ldots,i_p} \to \mathbb{P}^r_k$ is the inclusion, we note that the restriction functor i^{-1} commutes with colimits since it admits a left adjoint $i_!$ (the extension by zero functor). Also, the "tilde" functor commutes with colimits since it is an equivalence of categories. Hence, we obtain

$$\begin{aligned}\left(\bigoplus_{n\in\mathbb{Z}}\mathcal{O}(n)\right)\Big|_{U_{i_0\ldots i_p}} &= \bigoplus_{n\in\mathbb{Z}}\mathcal{O}(n)|_{U_{i_0\ldots i_p}}\\ &= \bigoplus_{n\in\mathbb{Z}}(\widetilde{S_{x_{i_0}\cdots x_{i_p}}})_n\\ &= \widetilde{S_{x_{i_0}\cdots x_{i_p}}}.\end{aligned}$$

Now let $p : U_{i_0,\ldots,i_p} \to \operatorname{Spec} k$ be the structure morphism. Since $\Gamma(U_{i_0,\ldots,i_p},-) = p_*$ can be identified with the forgetful functor at the level of modules, we see that it commutes with coproducts. Hence, the sections over $U_{i_0,\ldots,i_p}$ acquire a canonical grading, and, applying $\Gamma(U_{i_0,\ldots,i_p},-)$ to the above, we obtain the desired isomorphism of graded k-algebras.

(4) (B) Prove that $H^i(\mathbb{P}^r_k,\mathcal{O}(d)) = 0$ for $0 < i < r$. Use induction on r.

For $r > 1$, use the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r_k}(-1) \xrightarrow{\cdot x_r} \mathcal{O}_{\mathbb{P}^r_k} \to i_*\mathcal{O}_H \to 0,$$

where $H := Z(x_r)$ and $i : H \hookrightarrow \mathbb{P}^r_k$ is the inclusion. (Note that this sequence is exact after tensoring over $\mathcal{O}_{\mathbb{P}^r_k}$ with the line bundle $\mathcal{O}(n)$, then use the long exact sequence on cohomology and the induction hypothesis).

Solution. If $r = 1$ there is nothing to prove, so let $r > 1$. Set $X := \mathbb{P}^r_k$. Using the projection formula and the fact that direct sums are exact, we obtain an exact sequence for each $d \in \mathbb{Z}$:

$$0 \to \mathcal{O}_X(d-1) \to \mathcal{O}_X(d) \to i_*\mathcal{O}_H(d) \to 0.$$

Since $i : H \to X$ is an affine morphism one has $R^j i_* = 0$ for all $j > 0$ and hence by the Leray spectral sequence we obtain $H^i(X, i_*\mathcal{O}_H(d)) \simeq H^i(H,\mathcal{O}_H(d))$. But $H \cong \mathbb{P}^{r-1}_k$, and hence, by induction, we see $H^i(X,\mathcal{O}_H(d)) = 0$ for $0 < i < r-1$.

Passing to cohomology, we get a long exact sequence

$$\cdots \to H^i(X,\mathcal{O}_X(d-1)) \to H^i(X,\mathcal{O}_X(d)) \to H^i(X,\mathcal{O}_H(d)) \to \ldots$$

For $i = 0$ we claim that the sequence

$$0 \to H^0(X,\mathcal{O}_X(d-1)) \to H^0(X,\mathcal{O}_X(d)) \to H^0(H,\mathcal{O}_H(d)) \to 0$$

is exact. Indeed, this sequence is identified with the 0^{th} graded piece of the exact sequence of graded S-modules $0 \to S(-1) \to S \to S/(x_r) \to 0$. Hence, we see that the connecting map $\delta : H^0(H, \mathcal{O}_H(d)) \to H^1(X, \mathcal{O}_X(d-1))$ is zero.

By taking duals in the exact sequence

$$0 \to H^0(X, \mathcal{O}_X(-d-r-2)) \to H^0(X, \mathcal{O}_X(-d-r-1)) \to H^0(H, \mathcal{O}_H(-d-r-1)) \to 0$$

and using the residue pairing, we see that, at the other end of the long exact sequence, we have a short exact sequence

$$0 \to H^r(H, \mathcal{O}_H) \to H^r(X, \mathcal{O}_X(d)) \to H^r(X, \mathcal{O}_X(d-1)) \to 0.$$

Therefore the connecting map $\delta : H^{r-1}(X, \mathcal{O}_X(d-1)) \to H^r(H, \mathcal{O}_H)$ is zero, and we conclude that $H^i(X, \mathcal{O}_X(d-1)) \cong H^i(\mathcal{O}_X(d))$, induced by multiplication by x_r. If we now set $\mathcal{F} := \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X(d)$, by summing over $d \in \mathbb{Z}$ and using Question 3, we see that $H^i(X, \mathcal{F}(-1)) \simeq H^i(X, \mathcal{F})$, as S-modules, induced by multiplication by x_r.

Now let $U_r := \{x_r \neq 0\} \subseteq X$. By base change we have that that localization at x_r, $H^i(X, \mathcal{F})_{x_r}$ is isomorphic to $H^i(U_r, \mathcal{F}|_{U_r})$, which is zero for $i > 0$ by the easy part of Serre's criterion, as U_r is affine. In particular every element of $H^i(X, \mathcal{F})$ is annihilated by some power of x_r. Therefore we conclude that $H^i(X, \mathcal{F}) = 0$ for $0 < i < r$.

(5) (B) Let X be an integral Noetherian separated scheme, regular in codimension 1, and let f be a nonzero rational function on X. Prove that $\operatorname{div}(f)$ is in fact a Weil divisor, i.e., that the sum in the definition of $\operatorname{div}(f)$ is finite, not infinite.

Solution. Let us recall the definition of $\operatorname{ord}_Z(f)$ Let Z be a prime divisor with generic point η. The assumptions on X implies that all local rings $\mathcal{O}_{X,\eta}$ are Noetherian regular local rings of dimension 1 with quotient field K, the function field of X. In particular, $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with valuation $\operatorname{ord}_Z(g) := \operatorname{length}_{\mathcal{O}_{X,\eta}}(\mathcal{O}_{X,\eta}/g)$.

Hence, corresponding to Z we obtain a discrete valuation ord_Z on K: In particular $\operatorname{ord}_Z(f)$ is finite whenever $f \in K^\times$.

Let $U = \operatorname{Spec} A$ be any affine open subset of X such that $f \in \Gamma(U, \mathcal{O}_X^\times)$. Then any prime divisor $Z \subseteq X$ such that $\operatorname{ord}_Z(f) \neq 0$, is an irreducible component of $X \setminus U$. This is a Noetherian topological space, hence there can only be finitely many such Z.

(6) (B) Prove the "excision sequence" for the Weil class group. Let X be an integral Noetherian separated scheme, regular in codimension 1. Show that if $Z \subset X$ is an integral closed subscheme, with $\operatorname{codim} Z = 1$, and $U := X \setminus Z$ then the sequence

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl}(X) \to \operatorname{Cl}(U) \to 0$$

is exact. Deduce that if $U := \mathbb{P}_k^n \setminus$ (a degree d hypersurface), then $\operatorname{Cl}(U) \simeq \mathbb{Z}/d\mathbb{Z}$.

Solution. The map is induced by intersecting with U, i.e., $\sum n_i[Y_i] \mapsto \sum n_i[Y_i \cap U]$. It is well-defined since every rational function $f \in K(X)^\times$ can be viewed as a rational function $f \in K(U)^\times$. It is surjective, since if $Y \subset U$ is integral of codimension 1 in U, then $\overline{Y}$ is integral of codimension 1 in X and $\overline{Y} \cap U = Y$. The kernel of this map is given by those divisors with support along Z. Hence, we obtain the excision sequence.

If $Z \subseteq \mathbb{P}^n_k$ is a degree d hypersurface, then $[Z]$ is linearly equivalent to $d[H]$ where H is the hyperplane divisor $\{x_0 = 0\}$. Recalling that the choice of such a hyperplane divisor induces an isomorphism $\mathrm{Cl}(\mathbb{P}^n_k) \simeq \mathbb{Z}$, we see that $\mathrm{Cl}(U) \simeq \mathbb{Z}/d\mathbb{Z}$.

(7)(B) Let $X := Z(f) \subseteq \mathbb{P}^2_k$, where f is a degree d homogeneous equation such that $[1 : 0 : 0] \in \mathbb{P}^2_k \setminus X$; here $[x_0, x_1, x_2]$ are homogeneous coordinates on $\mathbb{P}^2_k$. Let $U_1 := X \cap \{x_1 \neq 0\}$ and $U_2 := X \cap \{x_2 \neq 0\}$.

a) Check that U_1 and U_2 are affine opens of X, and that X is separated.

b) Use the cover $\{U_1, U_2\}$ of X to compute that:
- $\dim H^0(X, \mathcal{O}_X) = 1$,
- $\dim H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}$.

Solution. (a) Noting that

$$U_1 \cong \operatorname{Spec} k[\frac{x_0}{x_1}, \frac{x_2}{x_1}]/(f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1})) \quad U_2 \cong \operatorname{Spec} k[\frac{x_0}{x_2}, \frac{x_1}{x_2}]/(f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)),$$

we see that U_1, U_2 are affine. X is separated since it is a closed subscheme of the separated scheme $\mathbb{P}^2_k$. This follows by considering the Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}^2_k & \xrightarrow{\Delta} & \mathbb{P}^2_k \times \mathbb{P}^2_k \end{array}$$

and using that closed immersions are stable under base change.

(b) The restriction maps induced by $U_{12} \to U_i$ are

$$\begin{aligned} k[\frac{x_0}{x_1}, \frac{x_2}{x_1}]/(f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1})) &\to k[\frac{x_0}{x_1}, \frac{x_2}{x_1}, \frac{x_1}{x_2}]/(f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1})), \\ k[\frac{x_0}{x_2}, \frac{x_1}{x_2}]/(f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)) &\to k[\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{x_2}{x_1}]/(f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)), \end{aligned} \tag{1}$$

the two rings on the left being isomorphic via $a(\frac{x_0}{x_1}, \frac{x_2}{x_1}) \mapsto a'(\frac{x_0}{x_2}, \frac{x_1}{x_2}) =: a(\frac{x_0}{x_2} \cdot (\frac{x_1}{x_2})^{-1}, (\frac{x_1}{x_2})^{-1})$. For convenience relabel $y_0 = \frac{x_0}{x_2}, y_1 = \frac{x_1}{x_2}$. Writing the polynomials in terms of their coefficients, we have $b(y_0, y_1) - a(y_0 y_1^{-1}, y_1^{-1}) = 0$ iff

$$\sum_{i,j \geq 0} \beta_{ij} y_0^i y_1^j - \sum_{i,j \geq 0} \alpha_{ij} y_0^i y_1^{-j-i} = 0, \tag{2}$$

which holds iff $\alpha_{ij} = \beta_{ij}$ whenever $i > 0$ or $j > 0$, and $\alpha_{00} = \beta_{00}$. Therefore $H^0(X, \mathcal{O}_X) = k$. The monomials which appear in the image are those of the form $y_0^i y_1^j$ with either ($i \geq 0$ and $j \geq 0$) or ($i \geq 0$ and $j \leq -i$). Note that we can write

$$f(y_0, y_1) = \sum_{\substack{i,j \geq 0 \\ i+j \leq d}} \varphi_{ij} y_0^i y_1^j. \tag{3}$$

By means of a projective linear transformation we may assume $[1, 0, 0] \in X$ and therefore the coefficient φ_{d0} of y_0^d is not 0. Therefore we may use f to eliminate any monomials $y_0^i y_1^j$ with $i \geq d$. Therefore the monomials appearing in the image are those such that

$$(j \geq 0 \vee j \leq -i) \wedge (0 \leq i \leq d), \tag{4}$$

whereas all monomials in the target are those with $0 \leq i \leq d$. Therefore we are missing a triangular region containing $\frac{1}{2}(d-1)(d-2)$ lattice points, so $\dim_k H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$.