Groups

ASO course Trinity 2024

Example sheet 2

1. Let A_∞ denote the even permutations of $\mathbb{N}$ which fix all but finitely many elements: that is A_∞ is the union

$$A_\infty = \bigcup_{n=1}^{\infty} A_n$$

where $A_n \subset A_{n+1}$ in the natural way. Show that A_∞ is an infinite simple group.

2. Let G be a group and $[G, G]$ its commutator subgroup (derived subgroup).

(i) Show that if $H \trianglelefteq G$ and G/H is Abelian then $[G, G] \leq H$.

(ii) Conversely, show that if $[G, G] \leq H \leq G$ then $H \trianglelefteq G$ and G/H is Abelian.

3. A sequence

$$\ldots \overset{\phi_{i-2}}{\to} G_{i-1} \overset{\phi_{i-1}}{\to} G_i \overset{\phi_i}{\to} \ldots$$

of groups and homomorphisms is called *exact* at G_i if

$$\ker \phi_i = \text{im } \phi_{i-1}$$

Show that if $N \trianglelefteq G$ then

$$1 \to N \to G \to G/N \to 1$$

is exact at N, G and G/N , where the middle two maps are inclusion and the canonical quotient map.

4. Verify Sylow's Theorems for the following groups

$$S_3,\ D_{12},\ A_4,\ S_4.$$

5. Let P be a non-trivial group of order p^m where p is prime (so P is a 'p-group').
By considering the conjugation action of P on itself prove that the centre

$$Z(P) = \{z \in P : zx = xz \text{ for all } x \in P\}$$

is nontrivial.
Deduce, using induction on m, that P is solvable. What can you say if $m = 2$?

6. Show that every group of order 350 is solvable.

7. Let G be a group of order 30.

(i) Show that either:

(1) there is a normal subgroup N of order 5 and a subgroup H of order 3, or
(2) there is a normal subgroup N of order 3 and a subgroup H of order 5.

Deduce that G has a normal subgroup K isomorphic to C_{15}.

(ii) Let y be a generator of K and let x be an order 2 element. Show that

$$G = \{x^i y^j : 0 \leq i \leq 1,\ 0 \leq j \leq 14\}$$

(iii) Let $\psi \in \mathrm{Aut}(K)$ satisfy $\psi^2 = id_K$. Show that $\psi : y \mapsto y^i$ where $i \in \{1, 4, 11, 14\}$.

(iv) Deduce that there are exactly four groups of order 30, up to isomorphism.

8. Find all groups of order 2023.