B3.3 Algebraic Curves revision lecture, May 2024
To go over 2023 B3.3 paper

Dominic Joyce

These slides available on course webpage.

B3.3 2023 question 1

1(a)[6 marks] Let C be an algebraic curve in $\mathbb{CP}^2$. Define when a point $p \in C$ is *singular*, and if it is nonsingular define the *tangent line* T_pC. State the strong form of Bézout's Theorem, involving intersection multiplicities $I_p(C, D)$ (which you need not define). Give a necessary and sufficient condition for when $I_p(C, D) = 1$.

– All bookwork.

Let C be defined by polynomial $P(x, y, z)$. Then $p = [a, b, c]$ is a *singular point* of C if

$$P(a, b, c) = P_x(a, b, c) = P_y(a, b, c) = P_z(a, b, c) = 0.$$

Bézout's Theorem: Let C, D be algebraic curves in $\mathbb{CP}^2$ of degrees m, n with no common component. Then $\sum_{p \in C \cap D} I_p(C, D) = mn$. Learn this.

$I_p(C, D) = 1$ if and only if p is a nonsingular point of C and D and the tangent lines T_pC, T_pD are distinct. Learn this.

(b)[5 marks] Let C be an irreducible algebraic curve in $\mathbb{CP}^2$ of degree d, defined by a polynomial $P(x, y, z)$. By considering the intersection of C with the curve $\frac{\partial P}{\partial x} = 0$, show that C has at most $\frac{1}{2}d(d-1)$ singular points.

If $d = 1$ then $C \cong \mathbb{CP}^1$ is nonsingular, so suppose $d > 1$.
If $P_x = 0$ then $P = P(y, z)$ is a product of linear factors $\beta y + \gamma z$, contradicting C irreducible. So P_x is nonzero.
Let D be the curve $P_x = 0$. Then D has degree $d - 1$.
Note that C, D have no common component as C is irreducible of degree d, and $\deg D = d - 1 < d$. So Bézout applies, and $\sum_{p \in C \cap D} I_p(C, D) = d(d-1)$.
Now any singular point p of C lies in D as $P_x = P_y = P_z = 0$ at p. Also $I_p(C, D) \geqslant 2$ by the criterion. Hence

$$2(\#\text{singular points of } C) \leqslant \sum_{p \in C \cap D} I_p(C, D) = d(d-1),$$

and C has at most $\frac{1}{2}d(d-1)$ singular points.

(c)[5 marks] If $d > 1$, improve (b) to show that C has at most $\frac{1}{2}d(d-1) - 1$ singular points.
[*Hint: apply a projective transformation so that* $[1,0,0]$ *is a nonsingular point of* C.]

As C has only finitely many singular points by (b), it has a nonsingular point. After a projective transformation, suppose $[1,0,0]$ is a nonsingular point of C. Euler's relation gives

$$1.P_x(1,0,0) = dP(1,0,0) = 0.$$

Thus $[1,0,0]$ lies in $C \cap D$, and $I_{[1,0,0]}(C,D) \geqslant 1$, so as in (b)

$$2(\#\text{singular points of } C) + 1 \leqslant \textstyle\sum_{p \in C \cap D} I_p(C,D) = d(d-1).$$

As $d(d-1)$ is even, $2(\#\text{singular points of } C) + 2 \leqslant d(d-1)$, so C has at most $\frac{1}{2}d(d-1) - 1$ singular points.
Otherwise you'll get $\frac{1}{2}(d(d-1)-1)$, not what you want.

(d)[5 marks] Now let C be any algebraic curve of degree $d \geqslant 1$ in $\mathbb{CP}^2$, not necessarily irreducible, and write $C = C_1 \cup \cdots \cup C_k$, where the C_i are the irreducible components of C. Show that C has at most $\frac{1}{2}d(d-1)$ singular points.
[*Hint: observe that every singular point of C is either a singular point of some C_i, or an intersection point of two C_i, C_j for $i \neq j$.*]

Let C_i have degree d_i. Then $d = d_1 + \cdots + d_k$. Each C_i has at most $\frac{1}{2}d_i(d_i-1)$ singular points by (b). Also $C_i \cap C_j$ is at most $d_i d_j$ points (weak Bézout). So by the hint, C has at most

$$\begin{aligned}\sum_{i=1}^{k} \tfrac{1}{2}d_i(d_i-1) + \sum_{1\leqslant i<j\leqslant k} d_i d_j &= \tfrac{1}{2}(d_1+\cdots+d_k)(d_1+\cdots+d_k-1)\\ &= \tfrac{1}{2}d(d-1)\end{aligned}$$

singular points. Note: (b) does not apply as C is reducible.

(e)[4 marks] Briefly explain how to find examples of degree d curves C with exactly $\frac{1}{2}d(d-1)$ singular points for any $d \geqslant 1$.

Let C be the union of d generic projective lines $L_1, \ldots, L_d$. By genericness, can assume the points $L_i \cap L_j$ are distinct for $1 \leqslant i < j \leqslant d$. Then $\mathrm{Sing}(C) = \{L_i \cap L_j : 1 \leqslant i < j \leqslant d\}$ is $\binom{d}{2} = \frac{1}{2}d(d-1)$ points.

Note: this is the **only** way to get $\frac{1}{2}d(d-1)$ singular points. If any reducible component C_i of C has degree > 1, can combine (c),(d) to show that C has at most $\frac{1}{2}d(d-1) - 1$ singular points.

B3.3 2023 question 2

(a)[6 marks] Let C be a nonsingular algebraic curve in $\mathbb{CP}^2$ of degree d. Define a *point of inflection* of C. What is the maximum number of points of inflection that C can have, as a function of d? Justify your answer briefly.
[*You may assume that C and its Hessian curve have no common component.*]

– All bookwork. Let C be defined by polynomial $P(x, y, z)$. Let D be the *Hessian curve* defined by $\det \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix} = 0$. A point of inflection is a (nonsingular) point of C which lies in D.
If $d = 1$ then every point of C is a point of inflection, as $P_{xx} = 0$, etc.
If $d = 2$ then the matrix above is constant and invertible (as C nonsingular), so no points of inflection.
If $d > 2$ then C has at most $3d(d-2)$ points of inflection by weak Bézout, as D has degree $3(d-2)$. Remember to cover all 3 cases.

(b)[5 marks] Let C be the nonsingular cubic curve in $\mathbb{CP}^2$ defined by the equation

$$x^3 + y^3 + 3xz^2 = 0.$$

Find all the points of inflection of C.

Hessian curve is

$$0 = \det \begin{pmatrix} 6x & 0 & 6z \\ 0 & 6y & 0 \\ 6z & 0 & 6x \end{pmatrix} = 216(x^2y - yz^2) = 216y(x-z)(x+z).$$

Inflection points are (i) $y = 0$ and $x^3 + y^3 + 3xz^2 = 0$: [0,0,1], $[\pm\sqrt{3}i, 0, 1]$ (3 points).
(ii) $x = z$ and $x^3 + y^3 + 3xz^2 = 0$: $[1, \sqrt[3]{-4}, 1]$ (3 points).
(iii) $x = -z$ and $x^3 + y^3 + 3xz^2 = 0$: $[1, \sqrt[3]{-4}, -1]$ (3 points).
Sanity check: 9 points of inflection, consistent with (a). Every nonsingular cubic has 9 points of inflection.

(c)[7 marks] Show that C in part (b) can be taken by a projective transformation to a cubic C_λ of the form

$$y^2z - x(x-z)(x-\lambda z) = 0, \qquad (1)$$

for some $\lambda \in \mathbb{C} \setminus \{0,1\}$ which you should determine.

Notes: partly bookwork, we follow the proof of the theorem in lectures on normal form of a cubic. You have to start by choosing an inflection point; life is easier if we choose the simplest $[0,0,1]$.

Step 1. Choose a point of inflection, apply projective transformation so it is $[0,1,0]$ with tangent line $z=0$.

We know $[0,0,1]$ is a point of inflection, with tangent line $3x=0$. Apply projective transformation $[x,y,z] \mapsto [x',y',z']$ with $x=z'$, $y=x'$, $z=\sqrt{-\frac{1}{3}}y'$ (factor $\sqrt{-\frac{1}{3}}$ gives nicer answer).

This gives $P'(x',y',z') = z'^3 + x'^3 - y'^2z' = 0$. So $y'^2z' = (x'+z')(x'+e^{2\pi i/3}z')(x'+e^{-2\pi i/3}z') = (x'-az')(x'-bz')(x'-cz')$, $a=-1$, $b=-e^{2\pi i/3}$, $c=-e^{-2\pi i/3}$.

Step 2. Apply projective transformation (standard from notes) $[x',y',z'] \mapsto [x'',y'',z'']$ with $x''=\frac{x'-az'}{b-a}$, $y''=(b-a)^{-3/2}y'$, $z''=z'$. Then

$$P''(x'',y'',z'')=(b-a)^{-3}\big(y''^2z''-x''(x''-z'')(x''-\lambda z'')\big)$$

for $\lambda=\frac{c-a}{b-a}=\frac{e^{-2\pi i/3}-1}{e^{2\pi i/3}-1}=\frac{1}{2}+\frac{i\sqrt{3}}{2}$. This is what you want.
Answer is not unique; as in (d), could have got several different answers, depending on order chosen for a,b,c.

(d)[7 marks] Show that, for general $\lambda \in \mathbb{C} \setminus \{0,1\}$, the cubic C_λ in (c) may be taken to a different curve $C_{\tilde\lambda}$ by a projective transformation with matrix of the form

$$\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix}, \qquad (2)$$

and find the possibilities for $\tilde\lambda$ as a function of λ (there are six, including $\tilde\lambda = \lambda$).

Let (2) map $(\tilde x\, \tilde y\, \tilde z)^T$ to $(x\, y\, z)^T$, so that $x = a\tilde x + b\tilde z$, $y = c\tilde y$, $z = d\tilde z$. This turns the polynomial in (1) into

$$c^2 d\tilde y^2 \tilde z - a^3(\tilde x + \tfrac{b}{a}\tilde z)(\tilde x + \tfrac{b-d}{a}\tilde z)(\tilde x + \tfrac{b-\lambda d}{a}\tilde z).$$

To make this of the form (1), choose a, b, c, d so that $c^2 d = 1$, $a^3 = 1$, and $\{\frac{b}{a}, \frac{b-d}{a}, \frac{b-\lambda d}{a}\} = \{0, -1, -\tilde\lambda\}$. We can fix $a = 1$ and $c = d^{-1/2}$.

There are six possibilities, depending on the permutation of $\{0, -1, -\tilde\lambda\}$:

(i) $(b, b-d, b-\lambda d) = (0, -1, -\tilde\lambda)$: $b = 0, d = 1, \tilde\lambda = \lambda$.

(ii) $(b-d, b, b-\lambda d) = (0, -1, -\tilde\lambda)$: $b = -1, d = -1, \tilde\lambda = 1-\lambda$.

(iii) $(b, b-\lambda d, b-d) = (0, -1, -\tilde\lambda)$: $b = 0, d = \frac{1}{\lambda}, \tilde\lambda = \frac{1}{\lambda}$.

(iv) $(b, b-d, b-\lambda d) = (0, -1, -\tilde\lambda)$: $b = d = \frac{1}{\lambda-1}, \tilde\lambda = \frac{1}{1-\lambda}$.

(v) $(b-\lambda d, b, b-d) = (0, -1, -\tilde\lambda)$: $b = -1, d = -\frac{1}{\lambda}, \tilde\lambda = 1-\frac{1}{\lambda}$.

(vi) $(b-\lambda d, b-d, b) = (0, -1, -\tilde\lambda)$: $b = \frac{\lambda}{1-\lambda}, d = \frac{1}{1-\lambda}, \tilde\lambda = \frac{-\lambda}{1-\lambda}$.

Note: these act as a group of Möbius transformations on $\lambda \mapsto \tilde\lambda$, isomorphic to S_3. Can check your calculations by composing Möbius transformations and getting back one of the same 6.

(a)[7 marks] Let C be a nonsingular algebraic curve in $\mathbb{CP}^2$ of genus g. Define *divisors*, the *degree* of a divisor, *meromorphic differentials*, and *canonical divisors* on C. What is the degree of a canonical divisor? State the *Riemann–Roch Theorem*.
[*You may use the following notation without defining it: for a meromorphic function* $f : C \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, *and for a meromorphic differential* $f\,\mathrm{d}h$, *we write* (f) *and* $(f\,\mathrm{d}h)$ *for the associated divisors. For a divisor* D *on* C *we write* $\mathcal{L}(D)$ *for the set of meromorphic* $f : C \to \mathbb{CP}^1$ *with* $(f) + D \geqslant 0$, *together with* $f = 0$. *You may assume that* $\mathcal{L}(D)$ *is a finite-dimensional* $\mathbb{C}$*-vector space, and write* $\ell(D) = \dim_{\mathbb{C}} \mathcal{L}(D)$.]

All bookwork. κ canonical divisor, $\deg \kappa = 2g - 2$. Learn this.
Riemann–Roch: D divisor, κ canonical divisor, then

$$\ell(D) - \ell(\kappa - D) = \deg D + 1 - g.$$

Learn this.

(b)[5 marks] Write $\mathrm{HD}(C)$ for the vector space of *holomorphic differentials* (i.e. meromorphic differentials with no poles) on C. Prove that $\dim \mathrm{HD}(C) = g$.

All bookwork. Let ω be a meromorphic differential, and $\kappa = (\omega)$ its canonical divisor. Any other meromorphic differential $\tilde{\omega}$ may be written $\tilde{\omega} = f\omega$ for f meromorphic. Then $\tilde{\omega}$ is holomorphic iff $(\tilde{\omega}) = (f) + (\omega) = (f) + \kappa \geqslant 0$, that is, iff $f \in \mathcal{L}(\kappa)$. (Something is holomorphic iff its divisor is nonnegative, i.e. it has zeroes but not poles.) So mapping $f \mapsto f\omega$ gives an isomorphism $\mathcal{L}(\kappa) \to \mathrm{HD}(C)$, and $\dim \mathrm{HD}(C) = \ell(\kappa)$.
Riemann–Roch with $D = 0$ gives

$$\ell(0) - \ell(\kappa) = 1 - g.$$

But $\mathcal{L}(0)$ is the vector space of holomorphic functions $f : C \to \mathbb{C}$, which are constant by the maximum principle, so $\mathcal{L}(0) = \mathbb{C} \cdot 1$, and $\ell(0) = 1$. (Learn this.) Hence $\dim \mathrm{HD}(C) = \ell(\kappa) = g$.

(c)[4 marks] Now let $g = 1$, so that $\mathrm{HD}(C) = \langle f\,\mathrm{d}h\rangle_{\mathbb{C}}$ by (b). Prove that $f\,\mathrm{d}h$ has no zeroes.

Let $(f\,\mathrm{d}h) = \kappa$. Then $\kappa \geqslant 0$ as $f\,\mathrm{d}h$ has no poles. But $\deg \kappa = 2g - 2 = 2 - 2 = 0$, so $\kappa = 0$ (as it can't have zeroes but no poles and still have degree 0). Hence $f\,\mathrm{d}h$ has no zeroes.

Note: $g = 1$ means that as a Riemann surface C is a torus $\mathbb{C}/\Lambda$. Then $\mathrm{d}w$ is a nonvanishing holomorphic differential on C, where w is the coordinate on $\mathbb{C}$; here $\mathrm{d}w$ is invariant under translation by Λ, so descends from $\mathbb{C}$ to $\mathbb{C}/\Lambda$. So it is not surprising that a genus 1 curve should have nonvanishing holomorphic differentials.

(d)[4 marks] Let $\Lambda \subset \mathbb{C}$ be a lattice, and $\wp(w)$ be the associated Weierstrass $\wp$-function. You may assume that $\wp$ satisfies $\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ for distinct $e_1, e_2, e_3 \in \mathbb{C} \setminus \{0\}$, and that the map

$$\Phi : \mathbb{C}/\Lambda \longrightarrow \mathbb{CP}^2, \qquad \Phi : w + \Lambda \longmapsto \begin{cases} [\wp(w), \wp'(w), 1], & w \notin \Lambda, \\ [0, 1, 0], & w \in \Lambda, \end{cases}$$

defines an isomorphism of Riemann surfaces from $\mathbb{C}/\Lambda$ to the nonsingular cubic curve C with equation

$$y^2 z = 4(x - e_1 z)(x - e_2 z)(x - e_3 z).$$

Write down an explicit nonzero holomorphic differential on C, in terms of the homogeneous coordinates x, y, z on $\mathbb{CP}^2 \supset C$, with brief justification.

We have $\wp(w) = \frac{x}{z}$ and $\wp'(w) = \frac{y}{z}$. So try $\omega = \frac{z}{y} \cdot \mathrm{d}\frac{x}{z}$ as the meromorphic differential. In terms of the local coordinate w on C we have

$$\omega = (\wp'(w))^{-1}\mathrm{d}(\wp(w)) = (\wp'(w))^{-1} \cdot \wp'(w)\mathrm{d}w = \mathrm{d}w,$$

which has no zeroes or poles.

This is motivated by the previous comment that $\mathrm{d}w$ is a nonvanishing holomorphic differential on $\mathbb{C}/\Lambda$. The trick for this part was to work out how to write $\mathrm{d}w$ in terms of $\wp(w)$ and $\wp'(w)$, as the x, y, z coordinates are $\wp(w), \wp'(w), 1$.

(e)[5 marks] Suppose that $g = 1$. Show that for generic choices of points $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ in C for $k > 0$, there does not exist a meromorphic function $f : C \to \mathbb{CP}^1$ with degree 1 zeroes at $p_1, \ldots, p_k$, degree 1 poles at $q_1, \ldots, q_k$, and no other zeroes or poles.
[*Hint: compute* $\ell(q_1 + \cdots + q_k)$.]

Riemann–Roch gives

$$\ell(q_1 + \cdots + q_k) - \ell(\kappa - q_1 - \cdots - q_k) = k + 1 - g = k.$$

Now $\deg(\kappa - q_1 - \cdots - q_k) = 2g - 2 - k = -k < 0$.
Useful fact: if $\deg D < 0$ then $\mathcal{L}(D) = \ell(D) = 0$. This holds as if $0 \neq f \in \mathcal{L}(D)$ then $\deg f = 0$, so $\deg f + \deg D < 0$, which contradicts $(f) + D \geqslant 0$, condition for $0 \neq f \in \mathcal{L}(D)$. Learn this.
Thus $\ell(\kappa - q_1 - \cdots - q_k) = 0$ and $\ell(q_1 + \cdots + q_k) = k$, for $k > 0$.
Hence $\mathbb{P}(\mathcal{L}(q_1 + \cdots + q_k)) \cong \mathbb{CP}^{k-1}$, which has dimension $k - 1$.

The set of zeroes of $0 \neq f \in \mathcal{L}(q_1 + \cdots + q_k)$ depends only on $[f] \in \mathbb{P}(\mathcal{L}(q_1 + \cdots + q_k))$. Thus there can only be a $(k-1)$-dimensional family of sets of points $(p_1, \ldots, p_k)$ that are the zeroes of $0 \neq f \in \mathcal{L}(q_1 + \cdots + q_k)$. But the family of all choices of $(p_1, \ldots, p_k)$ is k-dimensional, where $k > k-1$, so a generic choice of $(p_1, \ldots, p_k)$ cannot correspond to $0 \neq f \in \mathcal{L}(q_1 + \cdots + q_k)$.