Review of course

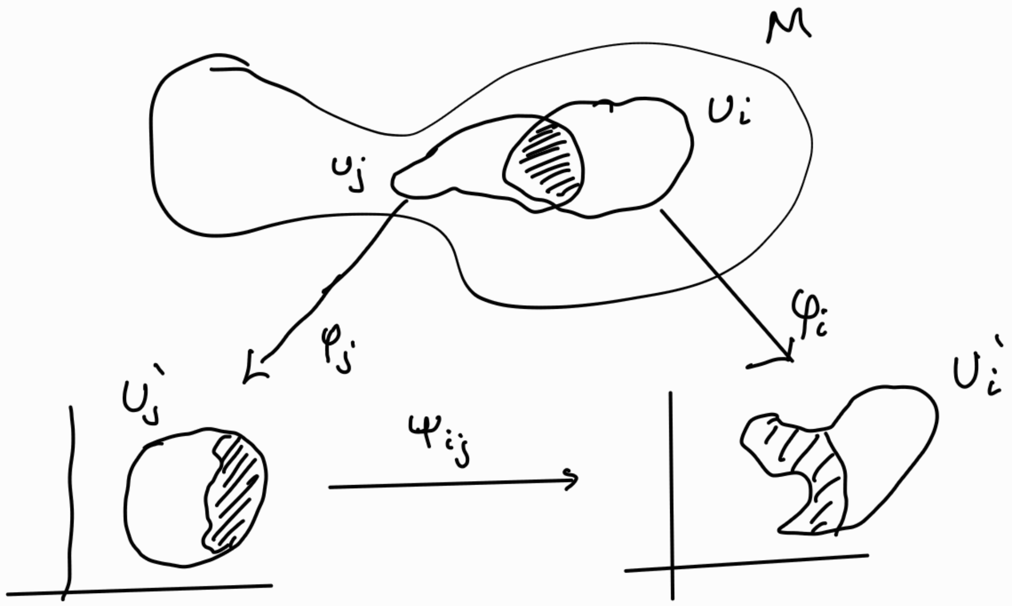
Manifolds

M an n-dim differentiable manifold if it satisfies

1) M is a Hausdorff topological space
2) M is provided with a family of pairs $\{(U_i, \varphi_i)\}$
3) $\{U_i\}$ a family of open sets which covers M: $\bigcup_i U_i = M$
4) φ_i a homeomorphism from U_i onto an open subset U_i' of $\mathbb{R}^n$
5) Given U_i and U_j s.t. $U_i \cap U_j \neq \emptyset$, then $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$

from $\varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$ is ∞-ly differentiable

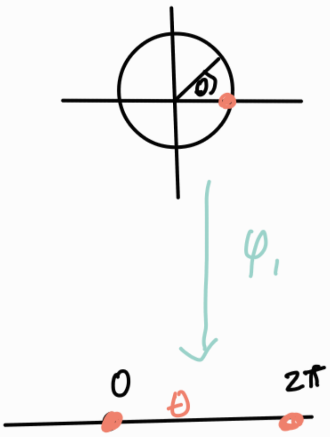
ψ_{ij} called transition functions

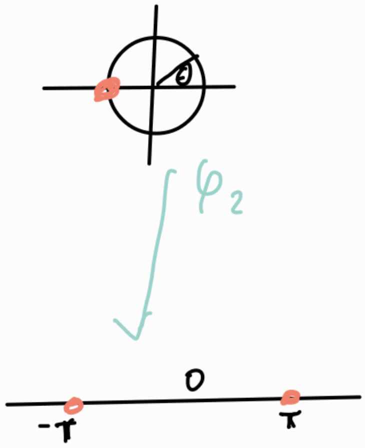


E.G.

S^1

$x^2+y^2=1$ in $\mathbb{R}^2$





$\varphi_1^{-1} : (0,2\pi) \longrightarrow S^1$

$\varphi_1^{-1}(\theta) \Rightarrow (\cos\theta, \sin\theta)$

$\mathrm{Im}\,\varphi_1^{-1} = S^1 - (\{1,0\})$

$\varphi_2^{-1} : (-\pi,\pi) \longrightarrow S^1$

$\varphi_2^{-1}(\theta) = (\cos\theta, \sin\theta)$

$\mathrm{Im}\,\varphi_2^{-1} = S^1 - \{(-1,0)\}$

$$\varphi_2(\varphi_1^{-1}(\theta)) = \begin{cases} \theta & \theta \in (0,\pi) \\ \theta - 2\pi & \theta \in (\pi, 2\pi) \end{cases}$$

not defined at 0 or π

See it is smooth everywhere

Tangent vectors

take a curve $\gamma : (a,b) \longrightarrow M$ and $f : M \longrightarrow \mathbb{R}$

let $t \in (a,b)$. We can define the directional derivative of

f along the curve at $t = t_*$ to be:

$$\left.\frac{d f(\gamma(t))}{d t}\right|_{t=t_*} \qquad t_* \in (a,b)$$

In local coordinates

$$\frac{\partial(f \circ \varphi^{-1})}{\partial x^\mu} \left.\frac{d x^\mu(\gamma(t))}{dt}\right|_{t=t_*}$$

Define $X = X^\mu \frac{\partial}{\partial x^\mu}$ $\quad : \quad X^\mu = \left.\frac{d x^\mu(\gamma(t))}{dt}\right|_{t=t_*}$

$$\left.\frac{d f(\gamma(t))}{dt}\right|_{t=t_*} = X^\mu \frac{\partial f}{\partial x^\mu} = X[f].$$

X is tangent vector to M at $p = \gamma(t_*)$ along the curve $\gamma(t)$.

Can identify an equivalence class of curves in M.

$\gamma_1(t) \sim \gamma_2(t)$ if 1) $\gamma_1(0) = \gamma_2(0) = p$

2) $\left.\frac{d x^\mu(\gamma_1(t))}{dt}\right|_{t=0} = \left.\frac{d x^\mu(\gamma_2(t))}{dt}\right|_{t=0}$

$$[\gamma(t)] = \left\{ \tilde{\gamma}(t) \;\middle|\; \gamma(0) = \tilde{\gamma}(0) = p \text{ and } \left.\frac{d x^\mu(\gamma)}{dt}\right|_{t=0} = \left.\frac{d x^\mu(\tilde{\gamma})}{dt}\right|_{t=0} \right\}$$

Identify tangent vector field X by equivalence classes.

All the tangent vectors at p form a vector space called the tangent space of M at p : $T_p(M)$

A basis is $\left\{ e^\mu = \frac{\partial}{\partial x^\mu} \right\}$

$$X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial y^\mu} = \tilde{X}^\mu \frac{\partial x^\nu}{\partial y^\mu} \frac{\partial}{\partial x^\nu}$$

$$\Rightarrow \quad \tilde{X}^{\mu} = X^{\nu} \frac{\partial y^{\mu}}{\partial x^{\nu}}$$

One-forms

Since $T_p(M)$ is a vector space $\exists$ a dual vector space $T_p^*(M)$ whose elements are linear maps from $T_p(M)$ to $\mathbb{R}$. $\quad V \in T_p(M)$

$$\langle df, V \rangle \equiv V(f) = V^{\mu} \frac{\partial f}{\partial x^{\mu}}$$

$\mathbb{R}$ linear in both V and f.

$\{dx^{\mu}\}$ as a basis

$$df = \frac{\partial f}{\partial x^{\mu}} dx^{\mu}$$

$$\omega = \omega_{\mu} dx^{\mu} = \tilde{\omega}_{\nu} dy^{\nu}$$

$$= \omega_{\mu} \frac{dx^{\mu}}{dy^{\nu}} dy^{\nu}$$

$$\Rightarrow \tilde{\omega}_{\nu} = \omega_{\mu} \frac{dx^{\mu}}{dy^{\nu}}$$

Tensor

Tensor of type (q, r) to be a multi-linear object which maps q elements of $T_p^*(M)$ and r elements of $T_p(M)$ $\rightarrow \mathbb{R}$

$$T = T^{\mu_1 \cdots \mu_q}{}_{\nu_1 \cdots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_q}} dx^{\nu_1} \cdots dx^{\nu_r}$$

What makes a tensor a tensor?

1) A multi-linear map:

$$T(\omega_1+\omega_2, v) = T(\omega_1, v) + T(\omega_2, v)$$
$$T(a\omega_1, v) = a\, T(\omega_1, v)$$

2) transformation under Lorentz

$$T'^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_L} = \frac{\partial y^{\mu_1}}{\partial x^{\sigma_1}} \cdots \frac{\partial y^{\mu_k}}{\partial x^{\sigma_k}} \frac{\partial x^{\tau_1}}{\partial y^{\nu_1}} \cdots \frac{\partial x^{\tau_1}}{\partial y^{\tau_1}} T^{\sigma_1 \dots \sigma_k}{}_{\tau_1 \dots \tau_L}$$

Differential forms

Totally anti-symmetrised tensors play a special role: differential forms.

Wedge product

$$dx^{\mu} \wedge \dots \wedge dx^{\mu_r} = \sum_{\sigma \in S_r} \operatorname{sign}(\sigma)\, dx^{\mu_{\sigma(1)}} \otimes \dots \otimes dx^{\mu_{\sigma(r)}}$$

Exterior derivative : $d\omega = \frac{1}{r!} \frac{\partial}{\partial x^{\mu}} \omega_{\nu_1 \dots \nu_r}\, dx^{\mu} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_r}$

Riemannian Geometry

Add extra structure : a metric

<u>Def</u> M a diff manifold

g a Riemannian metric on M if g a (0,2) tensor field

which at each point of p satisfies

$$g_p(X,Y) = g_p(Y,X)$$
$$g_p(X,X) \geq 0 \qquad = 0 \quad \text{iff } X = 0$$

Lorentzian manifold

Signature $(-1, +1, \dots, +1)$

$g(X,X) > 0$ X spacelike
$= 0$ null
< 0 time like

Connections

Affine connection is a map   vector fields

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$(X, Y) \longmapsto \nabla_X Y$ which satisfies

1) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$
2) $\nabla_{(fX + gY)} Z = f \nabla_X Z + g \nabla_Y Z$
3) $\nabla_X (fY) = X[f] Y + f \nabla_X Y$

By definition it takes 2 vectors to another vector

Define connection coefficents by $\left\{ \frac{\partial}{\partial x^\mu} \right\}$

$$\nabla_\nu = \nabla_{e_\nu} \qquad \nabla_\nu e_\mu = e^\lambda \Gamma^\lambda{}_{\nu\mu}$$

Then

$$\nabla_X Y = X^\mu \nabla_\mu (Y^\nu e_\nu)$$
$$= X^\mu \left\{ Y^\nu \nabla_\mu e_\nu + (\partial_\mu Y^\nu) e_\nu \right\}$$
$$= X^\mu \left\{ Y^\nu \Gamma^\lambda{}_{\mu\nu} e_\lambda + \partial_\mu Y^\lambda e_\lambda \right\}$$
$$= X^\mu \left[\partial_\mu Y^\lambda + Y^\nu \Gamma^\lambda_{\mu\nu} \right] e_\lambda$$

$\Rightarrow$

$$\underbrace{(\nabla_\nu Y)^\mu}_{\substack{\text{Sloppy notation, but accepted widely} \\ \nabla_\nu Y^\mu}} = \partial_\nu Y^\mu + \Gamma^\mu{}_{\nu\lambda} Y^\lambda$$

$\rightsquigarrow$ Can now apply this to other tensors

T a (q,r) tensor

$$\nabla_\mu T^{\nu_1 \cdots \nu_q}{}_{\rho_1 \cdots \rho_r} = \frac{\partial}{\partial x^\mu} T^{\nu_1 \cdots \nu_q}{}_{\rho_1 \cdots \rho_r} + \Gamma^{\nu_1}{}_{\mu\sigma} T^{\sigma \nu_2 \cdots \nu_q}{}_{\rho_1 \cdots \rho_r}$$
$$+ \cdots + \Gamma^{\nu_q}{}_{\mu\sigma} T^{\nu_1 \cdots \nu_{q-1} \sigma}{}_{\rho_1 \cdots \rho_r} - \Gamma^\sigma{}_{\mu\rho_1} T^{\nu_1 \cdots \nu_q}{}_{\sigma \cdots \rho_r} - \cdots - \Gamma^\sigma{}_{\mu\rho_r} T^{\nu_1 \cdots \nu_q}{}_{\rho_1 \cdots \sigma}$$

Connection transforms not as a tensor, rather it transforms so that $\nabla(\text{Tensor}) \rightsquigarrow \text{Tensor}$

Can define some new tensors

Torsion:

$$T(\omega; X, Y) = \omega\left(\nabla_X Y - \nabla_Y X - [X, Y] \right)$$

$$\omega \in \Omega^1(M) \qquad X, Y \in \mathfrak{X}(M)$$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}$$

Curvature:

$$R(\omega; X, Y, Z) = \omega\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \right)$$

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\nu\rho}$$

symmetries.

$R_{\mu\nu\sigma\rho} = -R_{\mu\nu\rho\sigma}$ | $R_{\mu[\nu\rho\sigma]} = 0$

$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ | $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

Given a metric there is a special connection called the unique Levi-Civita connection s.t.

1) $\nabla_X g = 0$ metric compatible

2) $T = 0$

Parallel transport

$$X^\mu\big|_\gamma = \frac{dx^\mu(\lambda)}{d\lambda} \qquad \gamma \text{ a curve}$$

A tensor is parallel transported along γ if

$$\nabla_X T = 0$$

Affine
Geodesic satisfies

$$\nabla_X X = 0 \iff \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}$$

with X tangent to curve.

Can be constructed from the action

$$S = \int d\lambda \sqrt{\alpha g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu} \qquad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$\rightsquigarrow$ non-affine. Unless you pick an affine parameter

- timelike $\tau \sim$ proper time
- spacelike $s \sim$ proper length] null:

Einstein's equations

In a vacuum $R_{\mu\nu} = 0$

Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right)dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Birkhoff's thm

Schwarzschild is the unique spherically symmetric, asymptotically flat sol to Einstein in a vacuum.

Geodesics

$$\boxed{\text{E-L} \quad \mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

& constraint

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \epsilon$$

Conserved quantities: ignorable coordinates

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \Rightarrow \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = 0$$

For Schw. $\exists$ 2 useful conserved quantities

$$E = -\frac{\partial \mathcal{L}}{\partial \dot{t}} \qquad L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

S^2 symmetry $\Rightarrow$ start in $\theta = \frac{\pi}{2}$ plane

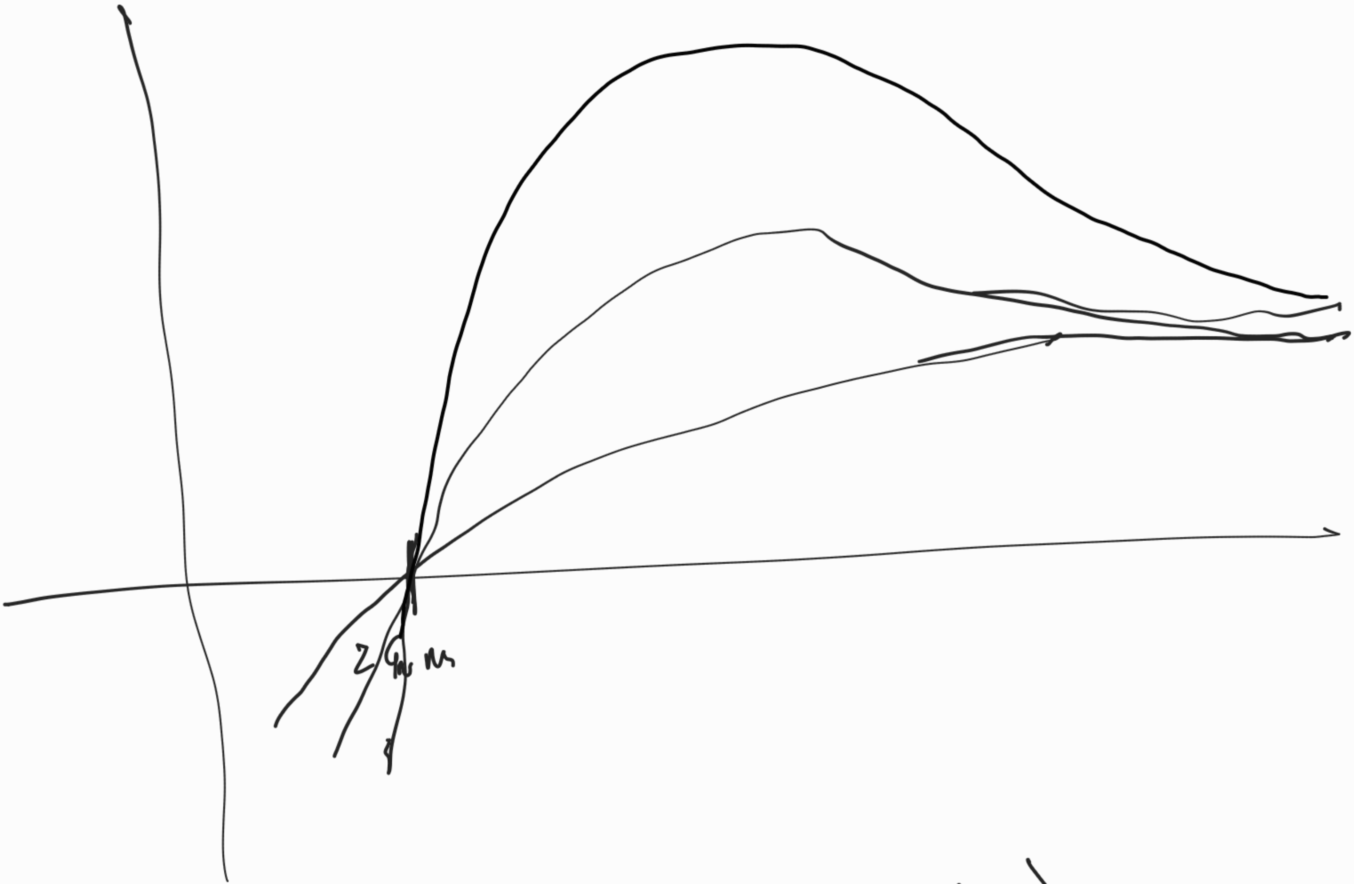
with $\dot{\theta}(\lambda=0) = 0$ then remains in plane.

$$\rightsquigarrow \epsilon = -\left(1 - \frac{2G_N M}{r}\right)^{-1} E^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} \dot{r}^2 + \frac{1}{r^2} L^2$$

$$\rightsquigarrow \frac{1}{2}\dot{r}^2 + V_{eff}(r) = \frac{E^2}{2}$$

$$V_{eff}(r) = -\frac{\epsilon}{2} + \frac{\epsilon G_N M}{r} + \frac{L^2}{2r^2}\left[1 - \frac{2G_N M}{r}\right]$$

Plot Massive



Circular orbits where $V'_{eff}(r_*) = 0$

$$\Rightarrow \dot{r}^2 = 0$$

Stable if minimum

unstable if max

As BH

Singularity at $r = 2G_N M$ $\qquad$ $r=0$ curvature

but coord sing

Change coords. Consider null radial $(\dot\theta = \dot\phi = 0)$ geodesics

$$0 = -f(r)\,\dot t^2 + f(r)^{-1}\,\dot r^2 \qquad f(r) = 1 - \frac{2G_N M}{r}$$

$$\frac{dt}{dr} = \pm\frac{1}{f(r)}$$

$$r_* = r + 2G_N M \log\left(\frac{r - 2G_N M}{2G_N M}\right)$$

$$t = \pm r_* + \text{const}$$

$$v = t + r_* \qquad u = t - r_* \quad \rightsquigarrow \text{ null radial geodesics are } u = \text{const} \text{ or } v = \text{const}$$

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dv^2 + 2\,dv\,dr + r^2 ds^2(S^2)$$

extend past horizon @ $r = 2G_N M$

Kruskal
$$U = -\exp\left(-\frac{u}{4G_N M}\right) \qquad V = \exp\left(\frac{v}{4G_N M}\right)$$

Outside horizon $\quad U < 0 \quad V > 0$

$$UV = \frac{2G_N M - r}{2G_N M}\exp\left(\frac{r}{2G_N M}\right)$$

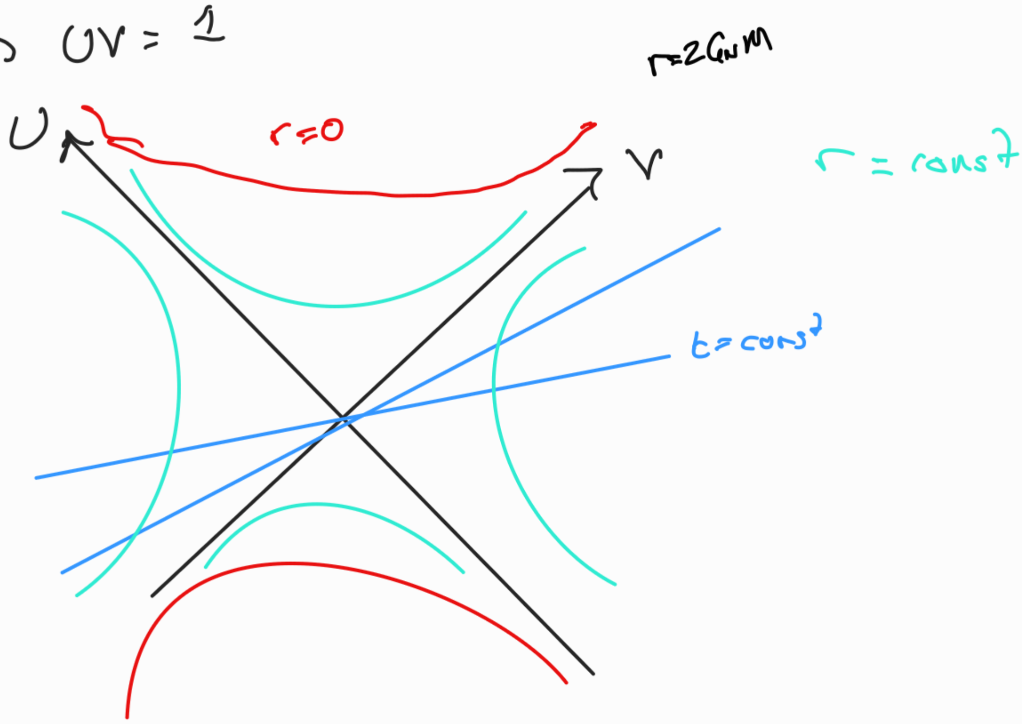
$$\frac{U}{V} = -\exp\left(-\frac{t}{2G_N M}\right)$$

Original Sch. covers only $U<0$ $V>0$ extended to $U, V \in \mathbb{R}$

$$ds^2 = -\frac{32(G_N M)^3}{r} e^{-\frac{r}{2G_N M}} dU dV + r^2 d\Omega(S^2)$$

$r = 2G_N M \Rightarrow U=0$ or $V=0$

$r=0 \Rightarrow UV = 1$



Cosmology

Friedmann - Robertson - Walker metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr}{1-kr^2} + r^2 d\Omega(S^2) \right)$$

scale factor

leads to cosmo-redshift

$\rightsquigarrow$ Friedmann eqs $\hat{=}$ Einstein

$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$ $\quad U^\mu = (1,0,0,0)$

Impose eq of state

$$p = \omega \rho$$

$$\rightsquigarrow \quad \frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}$$

$\rightsquigarrow$ Eqs to solve

Different behaviours depending on

- eq of state
- cosmological const
- $k \sim$ flat, closed, open universe

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