C3.3 Differentiable Manifolds revision lecture, May 2024
To go over 2023 C3.3 paper

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These slides available on course website (a)[6 marks] Define a *chart*, an *atlas*, and a *maximal atlas* on a topological space X. Define (*smooth*) *manifolds*.

All bookwork.
Don't forget Hausdorff and second countable conditions on X.

(b)[3 marks] Define X to be the set of unoriented affine real lines in $\mathbb{R}^3$, made into a topological space in the natural way. One way to do this is to note that

$$X \cong \big\{(\boldsymbol{u},\boldsymbol{v}) : \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^3 : |\boldsymbol{u}| = 1,\ \boldsymbol{u}\cdot\boldsymbol{v} = 0\big\}/(\boldsymbol{u},\boldsymbol{v}) \sim (-\boldsymbol{u},\boldsymbol{v}),$$

where $(\pm\boldsymbol{u},\boldsymbol{v})$ corresponds to the line $\big\{t\boldsymbol{u}+\boldsymbol{v} : t \in \mathbb{R}\big\}$. Prove that X has the properties required of the topological space of a manifold.

Need to show that X is Hausdorff and second countable.
The space $\big\{(\boldsymbol{u},\boldsymbol{v}) : \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^3 : |\boldsymbol{u}| = 1,\ \boldsymbol{u}\cdot\boldsymbol{v} = 0\big\}$ is both as it is a subset of $\mathbb{R}^6$ with the subspace topology, and $\mathbb{R}^6$ is both.
Hence X is both, as it is the quotient of a Hausdorff and second countable space by a finite group.

(c)[7 marks] Define three charts $(U_1,\varphi_1),(U_2,\varphi_2),(U_3,\varphi_3)$ on X by $U_1=U_2=U_3=\mathbb{R}^4$ and

$$\varphi_1:(a_1,b_1,c_1,d_1)\mapsto\big\{(x,y,z)\in\mathbb{R}^3: y=a_1x+b_1,\ z=c_1x+d_1\big\},$$
$$\varphi_2:(a_2,b_2,c_2,d_2)\mapsto\big\{(x,y,z)\in\mathbb{R}^3: z=a_2y+b_2,\ x=c_2y+d_2\big\},$$
$$\varphi_3:(a_3,b_3,c_3,d_3)\mapsto\big\{(x,y,z)\in\mathbb{R}^3: x=a_3z+b_3,\ y=c_3z+d_3\big\}.$$

Prove that $\big\{(U_1,\varphi_1),(U_2,\varphi_2),(U_3,\varphi_3)\big\}$ is an atlas on X. Deduce that X is a smooth manifold.
[*You may assume that* $(U_1,\varphi_1),(U_2,\varphi_2),(U_3,\varphi_3)$ *are charts.*]

Need to show the (U_i,φ_i) are pairwise compatible, and cover X.
The transition function $\varphi_2^{-1}\varphi_1$ maps

$$\varphi_2^{-1}\varphi_1:\big\{(a_1,b_1,c_1,d_1)\in\mathbb{R}^4:a_1\neq 0\big\}\to\big\{(a_2,b_2,c_2,d_2)\in\mathbb{R}^4:c_2\neq 0\big\},$$
$$\varphi_2^{-1}\varphi_1:(a_1,b_1,c_1,d_1)\mapsto\big(\tfrac{c_1}{a_1},d_1-\tfrac{b_1c_1}{a_1},\tfrac{1}{a_1},\tfrac{-b_1}{a_1}\big), \qquad (1)$$

as $y=a_1x+b_1,\ z=c_1x+d_1 \Leftrightarrow z=\frac{c_1}{a_1}y+(d_1-\frac{b_1c_1}{a_1}),\ x=\frac{1}{a_1}y-\frac{b_1}{a_1}$.

This is smooth, with smooth inverse

$$\varphi_1^{-1}\varphi_2 : \{(a_2,b_2,c_2,d_2)\in\mathbb{R}^4 : c_2\neq 0\} \to \{(a_1,b_1,c_1,d_1)\in\mathbb{R}^4 : a_1\neq 0\},$$
$$\varphi_1^{-1}\varphi_2 : (a_2,b_2,c_2,d_2) \mapsto \left(-\tfrac{1}{c_2}, -\tfrac{d_2}{c_2}, \tfrac{a_2}{c_2}, b_2 - \tfrac{a_2 d_2}{c_2}\right).$$

Hence (U_1,φ_1) and (U_2,φ_2) are compatible.
Similarly $(U_2,\varphi_2),(U_3,\varphi_3)$ and $(U_3,\varphi_3),(U_1,\varphi_1)$ are compatible, by cyclic permutation of $1,2,3$ and x,y,z.
A line in $\mathbb{R}^3$ lies in $\varphi_1(U_1),\varphi_2(U_2),\varphi_3(U_3)$ if it is not parallel to the (y,z) plane, or (x,z) plane, or (x,y) plane, respectively. As no line is parallel to all three,

$$X = \varphi_1(U_1)\cup\varphi_2(U_2)\cup\varphi_3(U_3).$$

Hence $\{(U_1,\varphi_1),(U_2,\varphi_2),(U_3,\varphi_3)\}$ is an atlas on X. It is contained in a unique maximal atlas.
We know X is Hausdorff and second countable by (b). Hence X is a smooth manifold.

(d)[6 marks] Prove that X is orientable.
[*Hint: prove the transition functions are orientation-preserving.*]

Differentiate $\varphi_2^{-1}\varphi_1$ in (1) at (a_1, b_1, c_1, d_1). It acts with matrix

$$D(\varphi_2^{-1}\varphi_1) = \begin{pmatrix} -\frac{c_1}{a_1^2} & 0 & \frac{1}{a_1} & 0 \\ \frac{b_1 c_1}{a_1^2} & -\frac{c_1}{a_1} & -\frac{b_1}{a_1} & 1 \\ -\frac{1}{a_1^2} & 0 & 0 & 0 \\ \frac{b_1}{a_1^2} & -\frac{1}{a_1} & 0 & 0 \end{pmatrix}$$

This has determinant $\frac{1}{a_1^4}$, as the only nonzero term comes from the product of the four red terms.
As $D(\varphi_2^{-1}\varphi_1)$ has positive determinant everywhere, $\varphi_2^{-1}\varphi_1$ is orientation-preserving. Similarly, $\varphi_3^{-1}\varphi_2$ and $\varphi_1^{-1}\varphi_3$ are orientation-preserving, by cyclic permutation of $1, 2, 3$ and x, y, z.
Hence $\{(U_1, \varphi_1), (U_2, \varphi_2), (U_3, \varphi_3)\}$ is an oriented atlas, and defines an orientation on X.

Note: we have several different ways to define orientations:

- as an orientation on $T_x X$ for $x \in X$, varying continuously with x.
- as an equivalence class $[\omega]$ of non-vanishing n-folds ω on X.
- as an atlas with orientation-preserving transition functions.

You can use any of these you like. This question uses the last.

(e)[3 marks] Now let Y be the set of (unoriented) affine real lines in $\mathbb{R}^2$, made into a manifold in a similar way. Is Y orientable? Give brief justification.

No, Y is not orientable, as it is topologically the Möbius strip, or equivalently $\mathbb{RP}^2 \setminus \{[1,0,0]\}$. (Space of projective lines in $\mathbb{RP}^2$ is $\mathbb{RP}^2$.)
[You can repeat the above calculations with two charts

$$\varphi_1 : (a_1, b_1) \mapsto \big\{(x,y) \in \mathbb{R}^2 : y = a_1 x + b_1\big\},$$
$$\varphi_2 : (a_2, b_2) \mapsto \big\{(x,y) \in \mathbb{R}^2 : x = a_2 y + b_2\big\}.$$

The transition function $\varphi_2^{-1}\varphi_1$ maps

$$\varphi_2^{-1}\varphi_1 : \big\{(a_1,b_1) \in \mathbb{R}^2 : a_1 \neq 0\big\} \to \big\{(a_2,b_2) \in \mathbb{R}^2 : a_2 \neq 0\big\},$$
$$\varphi_2^{-1}\varphi_1 : (a_1,b_1) \mapsto \left(\tfrac{1}{a_1}, -\tfrac{b_1}{a_1}\right).$$

We have $\det D(\varphi_2^{-1}\varphi_1) = \frac{1}{a_1^3}$, which changes sign at $a_1 = 0$ and is not orientation-preserving. This in itself doesn't prove Y not orientable, but going round the circle $b_1 = b_2 = 0$ in Y, you cross $a_1 = 0$ once, so orientations change sign around the circle.
This much detail not required.]

C3.3 2023 question 2

(a)[11 marks] (i) Let X be a manifold and $v \in \Gamma^\infty(TX)$ a vector field on X. Define the *maximal integral curve* of v through a point $x \in X$. What is the domain of a maximal integral curve if X is compact?
(ii) Define 1-*parameter groups of diffeomorphisms* $\varphi : \mathbb{R} \times X \to X$. In the case in which X is compact, describe the 1-1 correspondence between vector fields v and 1-parameter groups of diffeomorphisms φ, in terms of maximal integral curves.
(iii) If v is a vector field and α a tensor on X, define the *Lie derivative* $\mathcal{L}_v\alpha$.
[*You may assume the 1-1 correspondence in (ii) applies to* v.]

All bookwork.
For (iii), define $\mathcal{L}_v\alpha = \frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi_t^*(\alpha)\big)|_{t=0}$. If X is not compact then φ_t may not be defined if v is not complete – a 'local' definition is possible – but the question allows you to assume φ_t makes sense.

On $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3), define vector fields

$$u = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}, \qquad v = x_1^2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2} + x_3^2 \frac{\partial}{\partial x_3}.$$

(b)[4 marks] Find the maximal integral curves of u, v through each $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Write $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$. For γ to be a flow-line of u, need

$$\dot\gamma_1 = \gamma_1, \quad \dot\gamma_2 = \gamma_2, \quad \dot\gamma_3 = \gamma_3,$$

so $\gamma_i(t) = x_i e^t$. Domain of maximal integral curve is $\mathbb{R}$.

For γ to be a flow-line of v, need

$$\dot\gamma_1 = \gamma_1^2, \quad \dot\gamma_2 = \gamma_2^2, \quad \dot\gamma_3 = \gamma_3^2,$$

so $\int \frac{\mathrm{d}\gamma_i}{\gamma_i^2} = \int \mathrm{d}t$, and $-\frac{1}{\gamma_i} = t - \frac{1}{x_i}$, giving $\gamma_i(t) = \frac{x_i}{1-x_i t}$.
The domain of the maximal integral curve is (a, b), where

$$a = \begin{cases} -\infty, & \text{all } x_i \geqslant 0, \\ \max(\frac{1}{x_i} : x_i < 0), & \text{otherwise,} \end{cases}$$
$$b = \begin{cases} \infty, & \text{all } x_i \leqslant 0, \\ \min(\frac{1}{x_i} : x_i > 0), & \text{otherwise.} \end{cases}$$

(c)[6 marks] Prove that the only 2-form α on $\mathbb{R}^3$ with $\mathcal{L}_u\alpha = 0$ is $\alpha = 0$.
[*Well known formulae may be used if clearly stated.*]

Write $\alpha = \alpha_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \alpha_2 \mathrm{d}x_3 \wedge \mathrm{d}x_1 + \alpha_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2$ for $\alpha_i : \mathbb{R}^3 \to \mathbb{R}$ smooth. Cartan's formula: $\mathcal{L}_u\alpha = i_u(\mathrm{d}\alpha) + \mathrm{d}(i_u\alpha)$. So

$$
\begin{aligned}
\mathcal{L}_u\alpha &= i_u\Big[\Big(\tfrac{\partial\alpha_1}{\partial x_1} + \tfrac{\partial\alpha_2}{\partial x_2} + \tfrac{\partial\alpha_3}{\partial x_3}\Big)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3\Big] \\
&\quad + \mathrm{d}\big[\alpha_1 x_2 \mathrm{d}x_3 - \alpha_1 x_3 \mathrm{d}x_2 + \alpha_2 x_3 \mathrm{d}x_1 - \alpha_2 x_1 \mathrm{d}x_3 + \alpha_3 x_1 \mathrm{d}x_2 - \alpha_3 x_2 \mathrm{d}x_1\big] \\
&= \Big(\tfrac{\partial\alpha_1}{\partial x_1} + \tfrac{\partial\alpha_2}{\partial x_2} + \tfrac{\partial\alpha_3}{\partial x_3}\Big)\big(x_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 + x_2 \mathrm{d}x_3 \wedge \mathrm{d}x_1 + x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2\big) \\
&\quad + \Big(\tfrac{\partial\alpha_1}{\partial x_2}x_2 + \tfrac{\partial\alpha_1}{\partial x_3}x_3 + 2\alpha_1 - \tfrac{\partial\alpha_2}{\partial x_2}x_1 - \tfrac{\partial\alpha_3}{\partial x_3}x_1\Big)\mathrm{d}x_2 \wedge \mathrm{d}x_3 \\
&\quad + \Big(\tfrac{\partial\alpha_2}{\partial x_3}x_3 + \tfrac{\partial\alpha_2}{\partial x_1}x_1 + 2\alpha_2 - \tfrac{\partial\alpha_3}{\partial x_3}x_2 - \tfrac{\partial\alpha_1}{\partial x_1}x_2\Big)\mathrm{d}x_3 \wedge \mathrm{d}x_1 \\
&\quad + \Big(\tfrac{\partial\alpha_3}{\partial x_1}x_1 + \tfrac{\partial\alpha_3}{\partial x_2}x_2 + 2\alpha_3 - \tfrac{\partial\alpha_1}{\partial x_1}x_3 - \tfrac{\partial\alpha_2}{\partial x_2}x_3\Big)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\
&= \Big(\tfrac{\partial\alpha_1}{\partial x_1}x_1 + \tfrac{\partial\alpha_1}{\partial x_2}x_2 + \tfrac{\partial\alpha_1}{\partial x_3}x_3 + 2\alpha_1\Big)\mathrm{d}x_2 \wedge \mathrm{d}x_3 \\
&\quad + \Big(\tfrac{\partial\alpha_2}{\partial x_1}x_1 + \tfrac{\partial\alpha_2}{\partial x_2}x_2 + \tfrac{\partial\alpha_2}{\partial x_3}x_3 + 2\alpha_2\Big)\mathrm{d}x_3 \wedge \mathrm{d}x_1 \\
&\quad + \Big(\tfrac{\partial\alpha_3}{\partial x_1}x_1 + \tfrac{\partial\alpha_3}{\partial x_2}x_2 + \tfrac{\partial\alpha_3}{\partial x_3}x_3 + 2\alpha_3\Big)\mathrm{d}x_1 \wedge \mathrm{d}x_2.
\end{aligned}
$$

Thus $\mathcal{L}_u\alpha = 0$ provided

$$\frac{\partial\alpha_i}{\partial x_1}x_1 + \frac{\partial\alpha_i}{\partial x_2}x_2 + \frac{\partial\alpha_i}{\partial x_3}x_3 + 2\alpha_i = 0, \qquad i = 1,2,3.$$

Here is the tricky part:

Along the ray (tx_1, tx_2, tx_3) for $t \in \mathbb{R}$ this gives

$$t\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_i(tx_1, tx_2, tx_3)) + 2\alpha_i(tx_1, tx_2, tx_3) = 0,$$

with solution $\alpha_i(tx_1, tx_2, tx_3) = Ct^{-2}$.

But this is only continuous at $t = 0$ if $C = 0$, so when $t = 0$, $\alpha_i(x_1, x_2, x_3) = 0$. Thus $\alpha = 0$.

(d)[4 marks] Find all vector fields w on $\mathbb{R}^3$ with $\mathcal{L}_u w = 0$, that is, $[u, w] = 0$.
[*Well known formulae may be used if clearly stated.*]

Write $u = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}$ and $w = w_1 \frac{\partial}{\partial x_1} + w_2 \frac{\partial}{\partial x_2} + w_3 \frac{\partial}{\partial x_3}$.
Then $[u, w] = \sum_{i,j=1}^3 \big(u_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial u_j}{\partial x_i} \big) \frac{\partial}{\partial x_j}$. Learn this.
As $u_i = x_i$ we see that $[u, w] = 0$ iff

$$\frac{\partial w_i}{\partial x_1} x_1 + \frac{\partial w_i}{\partial x_2} x_2 + \frac{\partial w_i}{\partial x_3} x_3 - w_i = 0, \qquad i = 1, 2, 3.$$

(Another tricky part.) Along the ray (tx_1, tx_2, tx_3) for $t \in \mathbb{R}$ this gives

$$t \frac{\mathrm{d}}{\mathrm{d}t} (w_i(tx_1, tx_2, tx_3)) - w_i(tx_1, tx_2, tx_3) = 0,$$

with solution $w_i(tx_1, tx_2, tx_3) = Ct$. Thus w_i is linear along each ray in $\mathbb{R}^3$. For w_i to be smooth at $(0, 0, 0)$, this forces w_i to be linear, $w_i = \sum_{j=1}^3 a_{ij} x_j$. So the vector fields w with $\mathcal{L}_u w = 0$ are $w = \sum_{i,j=1}^3 a_{ij} x_j \frac{\partial}{\partial x_i}$ for real matrices $(a_{ij})_{i,j=1}^3$.

C3.3 2023 question 3

(a)[6 marks] Define the *de Rham cohomology groups* $H^k(X)$ of an n-manifold X. Show that if X is compact and oriented then there is a well-defined, surjective linear map $\Phi : H^n(X) \to \mathbb{R}$ with $\Phi([\omega]) = \int_X \omega$.
[*Standard results about integration of exterior forms may be used if clearly stated.*]
In the rest of the question you may assume that Φ is an isomorphism if X is connected.

All bookwork.
To show Φ is surjective, make an n-form ω with nonzero integral, supported in a small coordinate ball, using a 'bump function'.

(b)[5 marks] Let $f : X \to Y$ be a smooth map between compact, connected n-manifolds X, Y. Define the *degree* $\deg f$ of f, using de Rham cohomology. State an alternative definition in terms of preimages of points (you need not prove they are equivalent).

All bookwork.

(c)[9 marks] Show that the cohomology of $X = \mathcal{S}^2 \times \mathcal{S}^2$ may be written

$$H^0(X) = \langle 1_X \rangle_{\mathbb{R}}, \quad H^1(X) = 0, \;\; H^2(X) = \langle \alpha_1, \alpha_2 \rangle_{\mathbb{R}},$$
$$H^3(X) = 0, \;\; H^4(X) = \langle \alpha_1 \cup \alpha_2 \rangle_{\mathbb{R}},$$
$$\text{where} \quad \alpha_1 \cup \alpha_1 = 0, \quad \alpha_2 \cup \alpha_2 = 0, \quad \text{and} \quad \int_X \alpha_1 \cup \alpha_2 = 1.$$

[*You may assume the Künneth Theorem, and a formula for* $H^k(\mathcal{S}^2)$.]

Quote: $H^0(\mathcal{S}^2) \cong H^2(\mathcal{S}^2) \cong \mathbb{R}$, $H^1(\mathcal{S}^2) = 0$.
Künneth Theorem: $H^k(X \times Y) \cong \bigoplus_{i+j=k} H^i(X) \otimes H^j(Y)$, where the $H^i(X) \otimes H^j(Y)$ factor is the image of $\pi_X^*(H^i(X)) \cup \pi_Y^*(H^j(Y))$.
Write $H^0(\mathcal{S}^2) = \langle 1 \rangle_{\mathbb{R}}$ and $H^2(\mathcal{S}^2) = \langle \omega \rangle_{\mathbb{R}}$ with $\int_{\mathcal{S}^2} \omega = 1$. Write $\pi_1, \pi_2 : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2$ for the projections to first and second factors. Künneth says that $H^0(X) = \langle \pi_1^*(1) \cup \pi_2^*(1) \rangle_{\mathbb{R}} = \langle 1 \rangle_{\mathbb{R}}$, $H^1(X) = 0$, $H^2(X) = \langle \pi_1^*(1) \cup \pi_2^*(\omega) \rangle_{\mathbb{R}} \oplus \langle \pi_1^*(\omega) \cup \pi_2^*(1) \rangle_{\mathbb{R}} = \langle \pi_2^*(\omega), \pi_1^*(\omega) \rangle_{\mathbb{R}}$, $H^3(X) = 0$, and $H^4(X) = \langle \pi_1^*(\omega) \cup \pi_2^*(\omega) \rangle_{\mathbb{R}}$.

Set $\alpha_i = \pi_i^*(\omega)$. Then $H^2(X) = \langle \alpha_1, \alpha_2 \rangle_{\mathbb{R}}$, $H^4(X) = \langle \alpha_1 \cup \alpha_2 \rangle_{\mathbb{R}}$ as we want. Also $\alpha_1 \cup \alpha_1 = \pi_1^*(\omega) \cup \pi_1^*(\omega) = \pi_1^*(\omega \cup \omega) = 0$, as $\omega \cup \omega \in H^4(\mathcal{S}^2) = 0$. Similarly $\alpha_2 \cup \alpha_2 = 0$. And

$$\int_X \alpha_1 \cup \alpha_2 = \int_{\mathcal{S}^2 \times \mathcal{S}^2} \pi_1^*(\omega) \cup \pi_2^*(\omega) = \Bigl(\int_{\mathcal{S}^2} \omega\Bigr) \cdot \Bigl(\int_{\mathcal{S}^2} \omega\Bigr) = 1 \cdot 1 = 1.$$

(d)[5 marks] The cohomology of the compact oriented 4-manifold $Y = \mathbb{CP}^2$ may be written

$$H^0(Y) = \langle 1_Y \rangle_{\mathbb{R}}, \quad H^1(Y) = 0, \ \ H^2(Y) = \langle \beta \rangle_{\mathbb{R}},$$
$$H^3(Y) = 0, \ \ H^4(Y) = \langle \beta \cup \beta \rangle_{\mathbb{R}}, \ \ \text{where} \quad \int_Y \beta \cup \beta = 1.$$

Show that any smooth map $f : Y \to X$, with X defined as in (c), has degree $\deg f = 0$.

Write $f^*(\alpha_i) = a_i \beta$ for $i = 1, 2$. Then $f^*(\alpha_i \cup \alpha_i) = a_i^2 \beta \cup \beta$. But $\alpha_i \cup \alpha_i = 0$ and $\beta \cup \beta \neq 0$, so $a_i^2 = 0$, and $a_i = 0$.
Hence $f^*(\alpha_1 \cup \alpha_2) = a_1 a_2 \beta \cup \beta = 0$. The commuting diagram

$$\begin{tikzcd}
H^4(X) = \langle \alpha_1 \cup \alpha_2 \rangle_{\mathbb{R}} \arrow[r, "f^*"'] \arrow[d, "\cong"', "{[\lambda] \mapsto \int_X \lambda}"] & H^4(Y) = \langle \beta^2 \rangle_{\mathbb{R}} \arrow[d, "{[\lambda] \mapsto \int_Y \lambda}"', "\cong"] \\
\mathbb{R} \arrow[r, "\cdot \deg f"] & \mathbb{R}
\end{tikzcd}$$

now shows that $\deg f = 0$.