**Axiomatic Set Theory: Problem sheet 4**

**A.**

**1.** Prove 7.1.2, 7.1.3, and 7.1.6.

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7.1.2. states that the negation of a $\Sigma_0$ formula $\phi$ is logically equivalent to some $\Sigma_0$ formula $\phi^*$.

We define $\phi^*$ by recursion, noting that in each case it is $\Sigma_0$.

If $\phi$ is atomic, then let $\phi^* = \neg\phi$; this is $\Sigma_0$.

If $\phi = \neg\psi$, where $\psi$ is atomic, then let $\phi^* = \psi$; this is $\Sigma_0$.

If $\phi = \psi \wedge \chi$, then let $\phi^* = \psi^* \vee \chi^*$, and if $\phi = \psi \vee \chi$, then let $\phi^* = \psi^* \wedge \chi^*$; these are $\Sigma_0$.

If $\phi = \exists x \in y\, \psi$, then let $\phi^* = \forall x \in y\, \psi^*$, and if $\phi = \forall x \in y\, \psi$, then let $\phi^* = \exists x \in y\, \psi^*$; these are $\Sigma_0$.

7.1.3. states that $\Sigma_0$ formulae are absolute between transitive classes, and 7.1.6. states that $\Sigma_1$ formulae are upwards absolute between transitive classes.

Suppose that $U_1$ and $U_2$ are transitive classes. We prove that, for any $\Sigma_0$ formula $\phi(x_0, \ldots, x_n)$, and for any $a_0, \ldots, a_n \in U_1$, $\langle U_1, \in\rangle \vDash \phi(a_0, \ldots, a_n)$ if and only if $\langle U_2, \in\rangle \vDash \phi(a_0, \ldots, a_n)$ by induction on $\phi$.

We begin with atomic formulae.

The result is automatic for atomic formulae, and likewise for Boolean combinations of them.

Now we look at bounded quantification.

$\langle U_1, \in\rangle \vDash \exists x \in a_0\, \psi(a_1, \ldots, a_n, x)$ if and only if there exists $b \in U_1$ such that $\langle U_1, \in\rangle \vDash x \in a_0 \wedge \psi(a_1, \ldots, a_n, b)$, and this is if and only if $b \in U_1$, $b \in a_0$, and $\langle U_1, \in\rangle \vDash \psi(a_1, \ldots, a_n, b)$. Now $U_1$ is transitive, so $a_0 \subseteq U_1$, so this is equivalent to the statement that $b \in a_0$ and $\langle U_1, \in\rangle \vDash \psi(a_1, \ldots, a_n, b)$. By the inductive hypothesis, this is equivalent to that statement that $b \in a_0$ and $\langle U_2, \in\rangle \vDash \psi(a_1, \ldots, a_n, b)$, and reasoning similar to what we have already used tells us that this is equivalent to the statement that $\langle U_2, \in\rangle \vDash \exists x \in a_0\, \psi(a_1, \ldots, a_n, x)$.

Now suppose that $\phi(x_1, \ldots, x_n)$ is a $\Sigma_1$ formula, that $U_1 \subseteq U_2$ are transitive classes, that $a_1, \ldots, a_n$ are elements of $U_1$ and that $\langle U_1, \in\rangle \vDash \phi(a_1, \ldots, a_n)$. We argue by induction on the complexity of $\phi$ that $\langle U_2, \in\rangle \vDash \phi(a_1, \ldots, a_n)$.

The base cases and most instances of the inductive step are as above. The one extra thing we must do is deal with the case when $\phi(x_1, \ldots, x_n) = \exists x\, \psi(x_1, \ldots, x_n, x)$.

Suppose that $\langle U_1, \in\rangle \vDash \exists x\, \psi(a_1, \ldots, a_n, x)$. Then there is an element $b$ of $U_1$ such that $\langle U_1, \in\rangle \vDash \psi(a_1, \ldots, a_n, b)$. Now by the inductive hypothesis, $\langle U_2, \in\rangle \vDash \psi(a_1, \ldots, a_n, b)$, so $\langle U_1, \in\rangle \vDash \exists x\, \psi(a_1, \ldots, a_n, x)$.

From the assumption that $\Sigma_1$ formulae are upwards absolute between transitive classes, it follows that $\Pi_1$ formulae are downwards absolute.

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**2.** Prove 7.1.11 (30), ie. that "$x$ is a finite sequence of elements of $y$" (ie. $x \in {}^{<\omega}y$) is $\Sigma_0^{ZF}$, assuming that (1)–(29) of 7.11 are all $\Sigma_0^{ZF}$.

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We express this statement in the following way.

"$x$ is a function, there exists an element $z$ of $\omega$ such that $z = \operatorname{dom} x$, and for all $w \in \operatorname{ran} x$, $w \in y$."

We remove the reference to $\operatorname{ran} x$ as follows. Instead of saying "for all $w \in \operatorname{ran} x\, \psi$", we say: "for all $u \in x$, for all $v \in u$, for all $w \in v$, if there exists $v' \in u$ such that $w \notin v'$, then $\psi$" (the idea being that if $u \in x$, then $u$ is an ordered pair $\langle n, w\rangle$, and $\langle n, w\rangle = \{\{n\}, \{n, w\}\}$, so that $w$ belongs to just one element of $u$).

We have now expressed the statement in $\Sigma_0$.

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**B.**

**3.** Prove that "$x$ is a well-ordering of $y$" is $\Delta_1^{ZF}$.

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We express it first in $\Sigma_1$ and second in $\Pi_1$.

$\Sigma_1$: "$x$ is a relation on $y$ which is a total order (this is expressible in $\Sigma_0$), and there exists $z$ such that $z$ is an ordinal, and there exists $R$ such that $R$ is an order-isomorphism between $y$ and $z$."

$\Pi_1$: "$x$ is a relation on $y$ which is a total order (this is expressible in $\Sigma_0$), and for all $z$, if $z$ is a non-empty subset of $y$, then $z$ has a least element."

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**4.** Show that for every $\Sigma_1$ formula $\phi(x_1, \ldots, x_n)$, there exists a corresponding $\Sigma_0$ formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that

$$ZF \vdash \forall x_1, \ldots x_n(\phi(x_1, \ldots, x_n) \leftrightarrow \exists y_1, \ldots, y_m \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)).$$

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We do this by induction on $\phi$. This is trivial if $\phi$ is atomic, and easy for the cases when $\phi$ is a disjunction or a conjunction.

Suppose that $\phi(x_1, \ldots, x_n) = \exists x \in y\, \chi(x_1, \ldots, x_n, x)$, and that $\theta(x_1, \ldots, x_n, x, y_1, \ldots, y_n)$ is a $\Sigma_0$ formula such that $\exists y_1, \ldots, y_n\, \theta(x_1, \ldots, x_n, x, y_1, \ldots, y_n)$ is provably equivalent to $\chi(x_1, \ldots, x_n, x)$.

Then $\phi(x_1, \ldots, x_n)$ is provably equivalent to $\exists y_1, \ldots, y_n\, \exists x \in y\, \theta(x_1, \ldots, x_n, x, y_1, \ldots, y_n)$, which has the required form.

Now suppose that $\phi(x_1, \ldots, x_n) = \forall x \in y\, \chi(x_1, \ldots, x_n, x)$, and that $\theta(x_1, \ldots, x_n, x, y_1, \ldots, y_n)$ is a $\Sigma_0$ formula such that $\exists y_1, \ldots, y_n\, \theta(x_1, \ldots, x_n, x, y_1, \ldots, y_n)$ is provably equivalent to $\chi(x_1, \ldots, x_n, x)$.

Then $\phi(x_1, \ldots, x_n)$ is provably equivalent to $\exists f_1, \ldots, f_n\, \forall x \in y\, \big(f_i \text{isafunction}, \operatorname{dom} f_i = y\, \theta(x_1, \ldots, x_n, x, f_i(x), \ldots, f_n(x))\big)$, which has the required form.

Suppose that $\phi(x_1, \ldots, x_n) = \exists x\, \chi(x_1, \ldots, x_n, x)$, and that $\theta(x_1, \ldots, x_n, x, y_1, \ldots, y_n)$ is a $\Sigma_0$ formula such that $\exists y_1, \ldots, y_n\, \theta(x_1, \ldots, x_n, x, y_1, \ldots, y_n)$ is provably equivalent to $\chi(x_1, \ldots, x_n, x)$.

Then $\phi(x_1, \ldots, x_n)$ is provably equivalent to $\exists x \exists y_1, \ldots, y_n\, \theta(x_1, \ldots, x_n, x, y_1, \ldots, y_n)$, which has the required form.

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**5.** Prove that ordinal addition, multiplication and exponentiation are $\Delta_1^{ZF}$.

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These statements can be proved by repeatedly using Theorem 7.1.13.

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**6.** Prove that for any infinite cardinal $\kappa$, $\mathrm{cof}(\kappa)$ is a regular cardinal.

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Let $\mu = \mathrm{cof}(\kappa)$.

Suppose that $f : \mu \to \kappa$ is unbounded.

Define $g$ so that for each $\alpha < \kappa$, $g(\alpha) = \sup\{f(\beta) : \beta < \alpha\}$.

Then for all $\alpha < \kappa$, $g(\alpha) < \kappa$ also, or else $f \restriction \alpha$ has unbounded range in $\kappa$, contradicting minimality of $\mu$.

Clearly the range of $g$ is unbounded in $\kappa$.

Also $g$ is (non-strictly) monotonically increasing.

Now suppose that $\lambda \leq \mu$, $h : \lambda \to \mu$, and the range of $h$ is unbounded in $\mu$.

Then $g \circ h : \lambda \to \kappa$, and because $g$ is monotonically increasing, the range of $g \circ h$ is unbounded in $\kappa$.

Thus, by minimality of $\mu$, $\lambda = \mu$.

It follows at once that $\mathrm{cof}(\mu) = \mu$.

By considering the case when $h$ is a bijection, we see that $\mu$ must be a cardinal.

Thus $\mu$ is a regular cardinal.

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**7.** Suppose $\kappa, \lambda$ are infinite cardinals such that $\kappa \geq \lambda$. Prove that if $\lambda \geq \mathrm{cof}(\kappa)$, then $\kappa^\lambda > \kappa$. Suppose now that $\lambda < \mathrm{cof}(\kappa)$, and that $\kappa$ has the property that for any cardinal $\mu$, if $\mu < \kappa$ then $2^\mu \leq \kappa$. Prove that $\kappa^\lambda = \kappa$. Hence show that if GCH is assumed, then for any infinite cardinals $\kappa, \lambda$ with $\kappa \geq \lambda$, we have $\kappa^\lambda = \kappa$ or $\kappa^+$.

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Suppose that $\lambda \geq \mathrm{cof}(\kappa)$. Suppose, in order to obtain a contradiction, that $\kappa^\lambda \leq \kappa$. Let $\mu = \mathrm{cof}(\kappa)$. Then $\kappa^\mu \leq \kappa^\lambda$. Let $\langle g_\alpha : \alpha \in \kappa \rangle$ enumerate all functions from $\mu$ to $\kappa$. Let $f : \mu \to \kappa$ be a monotonic unbounded function (as in the solution to question 6.).

Now define $g : \mu \to \kappa$ so that for all $\alpha < \mu$, for all $\beta$ in the interval $[\sup\{f(\gamma) : \gamma < \alpha\}, f(\alpha))$, $g(\alpha) \neq g_\beta(\alpha)$; this is possible since $f(\alpha) < \mu = \mathrm{cof}(\kappa)$ so $\{g_\beta(\alpha) : \beta < f(\alpha)\}$ is not the entirety of $\kappa$.

Then for all $\beta$, $g \neq g_\beta$, giving a contradiction.

Suppose that $\lambda < \mathrm{cof}(\kappa)$, and that $\kappa$ has the property that for any cardinal $\mu$, if $\mu < \kappa$ then $2^\mu \leq \kappa$.

Suppose that $f : \lambda \to \kappa$. Then $\sup \mathrm{ran} f < \kappa$.

Hence there exists $\alpha < \kappa$ such that $f : \lambda \to \alpha$.

Note that the set of all functions $f$ such that $f : \lambda \to \alpha$ has cardinality $|\alpha|^\lambda$.

So $\kappa^\lambda \leq \sum_{\alpha<\kappa} |\alpha|^\lambda \leq \sum_{\alpha<\kappa} \kappa \leq \kappa.\kappa = \kappa$, so since $\kappa \leq \kappa^\lambda$, $\kappa^\lambda = \kappa$.

Now for any $\lambda \leq \kappa$, $\kappa^\lambda \leq \kappa^\kappa = 2^\kappa$. Under the assumption of GCH, we must therefore have $\kappa^\lambda \leq \kappa^+$. So $\kappa^\lambda$ is either $\kappa$ or $\kappa^+$.

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**C.**

**8.** Suppose $\kappa$ is an *uncountable regular* cardinal. Let $g : \kappa \to \kappa$ be any function. Prove that for any $\alpha < \kappa$, there exists $\beta < \kappa$, with $\alpha \leq \beta$, such that $\beta$ is closed under $g$ (ie. for all $\gamma < \beta$, $g(\gamma) < \beta$).

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Let $\alpha_0 = 0$, and given $\alpha_n$, let $\alpha_{n+1}$ be the supremum of $\{\alpha_n+1\} \cup \{f(\beta)+1 : \beta \leq \alpha_n\}$. $\kappa$ is regular so if $\alpha_n < \kappa$, then $\alpha_{n+1} < \kappa$ also.

Now let $\alpha_\omega = \sup_{n \in \omega} \alpha_n$. Then since $\kappa$ is uncountable and regular, $\alpha_\omega < \kappa$; and $\alpha_\omega$ is closed under $f$.

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**9.** Let $\kappa$ be an uncountable regular cardinal with the property that for any cardinal $\mu < \kappa$, we have $2^\mu < \kappa \ldots$ (*).

Prove that (i) if $\alpha$ is any cardinal and $\alpha < \kappa$, then $|V_\alpha| < \kappa$, (ii) $|V_\kappa| = \kappa$, (iii) if $\kappa$ is regular, then $\langle V_\kappa, \in \rangle \models$ ZFC.

(For (iii) you need consider only the replacement scheme, since we essentially showed that if $\alpha$ is a limit ordinal and $\alpha > \omega$, then $\langle V_\alpha, \in \rangle$ satisfies all the axioms of ZFC except, possibly, replacement.)

Deduce that in ZFC one cannot prove the existence of a cardinal that satisfies (*).

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(i) We prove that if $\alpha < \kappa$, then $|V_\alpha| < \kappa$, by inductino on $\alpha$.

If $\alpha = 0$, then $|V_0| = 0 < \kappa$.

If $\alpha = \beta + 1$, and $|V_\beta| < \kappa$, then $|V_{\beta+1}| = 2^{|V_\beta|} < \kappa$ by the property (*).

If $\lambda < \kappa$ is a limit, then $|V_\lambda| = |\bigcup_{\alpha<\lambda} V_\alpha| \leq \sum_{\alpha<\lambda} |V_\alpha|$. Now if $|V_\alpha| < \kappa$ for all $\alpha < \lambda$, then regularity of $\kappa$ gives that $|V_\lambda| < \kappa$ also.

(ii) $V_\kappa = \bigcup_{\alpha<\kappa} V_\alpha$, so $V_\kappa$ is a union of $\kappa$-many sets of size $< \kappa$. So $|V_\kappa| \leq \kappa$.

But also $\kappa \subseteq V_\kappa$, so $|V_\kappa| = \kappa$.

(iii) Suppose that $A \in V_\kappa$, and $F : A \to V_\kappa$ is a class term (definable by some formula $\phi(x, y)$ such that $\langle V_\kappa, \in \rangle \models \forall x \in A\, \exists! y\, \phi(x, y)$).

Then since $A \in V_\kappa$, $A \in V_\alpha$ for some $\alpha < \kappa$, so $A \subseteq V_\alpha$, so $|A| \leq |V_\alpha| < \kappa$.

Now apply Replacement in $V$ to $\phi^{V_\kappa}$; let $B = \{y : \exists x \in A\, \phi^{V_\kappa}(x, y)$. Note that $|B| \leq |A| < \kappa$.

For each $y \in B$, let $\gamma_y$ be least such that $y \in V_{\gamma_y}$; $\gamma_y < \kappa$ always.

Since $\kappa$ is regular, $\gamma = \sup\{\gamma_y : y \in B\} < \kappa$.

Then $B \subseteq V_\gamma$, so $B \in V_{\gamma+1} < V_\kappa$.

And $\langle V_\kappa, \in \rangle \models B = \{y : \exists x \in A\, \phi(x, y)\}$ as required.

Now suppose that from ZFC one could prove the existence of a cardinal having property (*). Then from ZFC one can prove the existence of an ordinal $\alpha$ such that $\langle V_\alpha, \in \rangle \models$ ZFC. Note that this statement is $\Sigma_0$ in parameters $V_\alpha$ and $\alpha$, and so is absolute.

Let $\alpha$ be least such that $\langle V_\alpha, \in \rangle \models$ ZFC. Then there does not exist $\beta < \alpha$ such that $\langle V_\beta, \in \rangle \models$ ZFC. Hence

$$\langle V_\alpha, \in \rangle \not\models \exists \beta\, \langle V_\beta, \in \rangle \models \text{ZFC},$$

giving a contradiction.

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