Revision lecture

Main areas of course (questions for the exam)

1. Approximation of integrals (IBPs, Laplace, MOSP, MOSD)
2. Matched asymptotic expansions (singular perturbations, BLs)
3. Method of multiple scales
4. WKB method

(Much of the initial part of the course §1 – §3 provides the background knowledge / expertise / understanding to use these techniques...)

① Asymptotic approximation of integrals

(i) Integration by parts

eg
$$I(x) = \int_x^\infty \tilde{v}(t)\,dt = \int_x^\infty u(t)\frac{d}{dt}v(t)\,dt = \underbrace{\Big[u(t)v(t)\Big]_x^\infty}_{\substack{\text{evaluate –}\\ \text{need } u(t)v(t)\\ \text{'sensible' as}\\ t\to\infty}} - \int_x^\infty \underbrace{v(t)\frac{d}{dt}u(t)\,dt}_{\substack{\text{want much}\\ \text{smaller than}\\ \text{original}\\ \text{integrand}}}$$

key point – doesn't want contributions from limits dominate wrt integral.

↳ can sometimes be fixed by writing eg as $\int_0^x = \int_0^\infty - \int_x^\infty$

– also doesn't want dominant contribution from an interior point

↳ Does give an explicit error term ✓

But, generally quite limited in applicability

(ii) Laplace's method

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\,dt \quad \text{as } x\to\infty$$

real, continuous functions.

key point – dominant contribution from region where $\phi(t)$ largest

↳ split into cases
1. max @ $t=a$,
2. max @ $t=c$ – interior point
3. max @ $t=b$

<u>Steps</u> – dominant contribution from region around max $\varphi(t)$

$\Rightarrow$ reduce domain of integration to around this region.

① expand $f(t)$, $\varphi(t)$ using Taylor series about max

② rescale the integration variable – to replace integration limits by ∞ (introducing only exponentially small errors)

To do – establish what 'small' means – using Taylor expansion of φ at max. (and need to consider whether eg $\varphi''(c)<0$ or $\varphi''(c)=0$ $\leftarrow$ here need different scaling)

$\rightarrow$ should be able to confidently evaluate how big the errors are – quantify them carefully and show they are small compared to the dominant term (see solution for the 2023 paper).

$\rightarrow$ should be able to write down solution for general $f(t)$ and $\varphi(t)$.

As part of this, often use the Watson's lemma:

$$I(x) = \int_0^b f(t) e^{-xt}\, dt \quad (b>0)$$

(If $b=\infty$, then we also need $f(t) \ll e^{ct}$ as $t\to\infty$ for some $c>0$ so integral also converges at $t=\infty$)

$f(t)$ cts with $f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n}$ as $t \to 0^+$ $\quad$ $\alpha>-1$, $\beta>0$ } so integral converges at $t=0$

Then, $I(x) \sim \sum_{n=0}^{\infty} \dfrac{a_n \Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}}$ as $x\to\infty$

<u>(iii) Method of stationary phase</u>

$$I(x) = \int_a^b f(t) e^{ix\psi(t)}\, dt$$

ie $\varphi(t) = i\psi(t)$ where $\psi(t)$ real

$\hookrightarrow$ purely imaginary $\Rightarrow$ behaves v. differently

as $x\to\infty$

Dominant contributions to the integral from regions where $\psi'(t)=0$ as there the oscillations don't cancel.

Will need the Riemann Lebesgue lemma: If $\int_a^b |f(t)|\,dt < \infty$ and $\psi(t)$ ctsly differentiable for $a\le t\le b$ and not constant on any subinterval in $a\le t\le b$ then $\int_a^b f(t) e^{ix\psi(t)}\,dt \to 0$ as $x\to\infty$.

Exemplar: $\psi'(c) = 0$ with $a<c<b$ and $\psi'(t) = 0$ o/w, with $f(c) \neq 0$ and $\psi''(t) \sim \text{ord}(1)$ in a neighbourhood of c.

– split integral into three – main contribution from region around c, so long as the ε (defining the region of integration) is sensibly chosen.

Method – Taylor expand $f(t)$, $\psi(t)$ about c
– change variables
– use contour integration

be confident where the approximations are made, and how big the errors are.

key points – errors only algebraically small (in Laplace's method they are exponentially small)
– very difficult to get higher order corrections – they come from the entire region of integration. (Laplace – the full asymptotic expansion here comes from the local region only)

(iv) Method of steepest descents

$$I(x) = \int_C f(t) e^{x\varphi(t)} dt$$

as $x \to \infty$

$t \in \mathbb{C}$
$x \in \mathbb{R}$
C – contour in the complex plane

(NB Laplace's method / method of stationary phase – special cases)

Key idea – deform to a new contour on which $\text{Im}\,\varphi(t)$ is piecewise constant and then evaluate on these contours using Laplace's method

Method – deform contour to a union of steepest descent (v = constant) contours through the end points and any relevant saddle points
– evaluate local contributions from saddles / end points using Laplace.

would have deformed to Γ s.t. $\text{Re}(\varphi)$ = constant to apply method of stationary phase – but we know that Laplace's method is better

generate all terms and tails are exponentially small.

Note that since local contributions dominate, we only need the tangents to the steepest descent paths to generate the asymptotic approximations.

often can save a lot of time and energy by noting this!

(v) Splitting the range of integration — split range of integration in order to use different approximations in each part.

② Matched asymptotic expansions — singular perturbation problems where setting $\varepsilon = 0$ reduces order of the differential equation.

Method — determine scaling of BLs (and their location)

- rescale independent variable in the BL
- find expansions in and outside of the BL
- determine the constants (matching + boundary conds.)
  ↳ matching either via Van Dyke's matching rule, ⊛ or using an intermediate variable that interpolates between the two (for suitable choice)

(⊛ Need to know when it might not work too)

③ Method of multiple scales — when there are two or more length or time scales in a differential equation
↳ arises through different processes having their own (different) timescales that act simultaneously.

Method — introduce multiple (usually two for the problems we consider) timescales. eg $\underbrace{\tau = t}_{\text{fast}}$ and $\underbrace{T = \varepsilon t}_{\text{slow}}$ $\Rightarrow \frac{d}{dt} \mapsto \frac{\partial}{\partial \tau} + \varepsilon \frac{\partial}{\partial T}$ etc

- Turns and ODE into a PDE
- Use an asymptotic expansion and check (and suppress) secular terms

④ WKB method — singular perturbation problems that do not have BLs.

Let $y(x) = e^{i\varphi(x)/\varepsilon} A(x;\varepsilon)$

will expand using an asymptotic expansion

do not expand $\varphi(x)$