Some Comments on Consistency Errors

Charlie Parker

1 Abstract definition

Consider a linear PDE posed on a domain $\Omega \subseteq \mathbb{R}^d$:

$$\mathcal{L}u = f \qquad \text{in } \Omega, \tag{1.1}$$

equipped with appropriate boundary and/or initial conditions. Note that Ω could be a spatial domain for a time independent problem or a space-time domain for time dependent problems. Let $\Omega_h \subseteq \Omega$ be the computational mesh, as defined in the lecture notes. Then, a finite difference scheme for (1.1) can be written abstractly as

$$\mathcal{L}_h U(\boldsymbol{x}) = F(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Omega_h, \tag{1.2}$$

where $\mathcal{L}_h$ is a linear finite difference operator and F is some approximation to f. Then, the consistency error Υ of the scheme (1.2) at a mesh point $\boldsymbol{x} \in \Omega_h$ is given by

$$\Upsilon(\boldsymbol{x}) := \mathcal{L}_h(u(\boldsymbol{x}) - U(\boldsymbol{x})) = \mathcal{L}_h e(\boldsymbol{x}) = \mathcal{L}_h u(\boldsymbol{x}) - F(\boldsymbol{x}), \tag{1.3}$$

where $e(\boldsymbol{x}) := u(\boldsymbol{x}) - U(\boldsymbol{x})$ is the (approximation) error. That is, the consistency error is simply the finite difference scheme applied to the error. In other words, the consistency error is what you get when you replace "U" with "u" and subtract F from the LHS. If $F(\boldsymbol{x}) = f(\boldsymbol{x})$, then an equivalent definition is

$$\Upsilon(\boldsymbol{x}) := \mathcal{L}_h u(\boldsymbol{x}) - \mathcal{L}u(\boldsymbol{x}).$$

Note that I have used Υ for the abstract definition since we use different letters for static vs time-dependent problems. We now look at a few examples from the notes.

2 Poisson equation

Let us now specialize to the usual setting where $\Omega = (0,1)^2$ and we have a uniform mesh (using all of the notation from the notes). Consider the Poisson equation

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

discretized with our favorite centered difference operator:

$$-\left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} \right) = f(x_i, y_j) \qquad 1 \le i, j \le N-1,$$

and $U_{0,j} = U_{N,j} = U_{i,0} = U_{i,N} = 0$ for $0 \le i, j \le N$. Then, $\mathcal{L}_h$ is given by

$$\mathcal{L}_h V_{i,j} := -\left(\frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h^2} + \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{h^2} \right), \qquad 1 \le i, j \le N-1,$$

and so the consistency error (denoted $\varphi_{i,j}$ for elliptic equations) is

$$\begin{aligned}\varphi_{i,j} &= \mathcal{L}_h(u(x_i, y_j) - U_{i,j}) \\ &= -\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}\right) - f(x_i, y_j)\end{aligned}$$

for $1 \leq i, j \leq N-1$, which should look familiar.

3 Heat equation

Let $\Omega = (-\infty, \infty) \times (0, T)$, where $T > 0$ is the final time and consider the heat equation:

$$\begin{aligned}\partial_t u - \partial_{xx} u &= 0 \qquad \text{in } \Omega, \\ u|_{t=0} &= u_0 \qquad \text{on } \mathbb{R}.\end{aligned}$$

For $\theta \in [0, 1]$, the θ-method is

$$\begin{aligned}\frac{U_j^{m+1} - U_j^m}{\Delta t} - \theta \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} - (1-\theta)\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} &= 0, \qquad j \in \mathbb{Z}, m = 0, \ldots, M-1, \\ U_j^0 &= u_0(x_j), \qquad j \in \mathbb{Z},\end{aligned}$$

where $\Delta t = T/M$. Then, $\mathcal{L}_h$ is given by

$$\mathcal{L}_h V_j^m := \frac{V_j^{m+1} - V_j^m}{\Delta t} - \theta \frac{V_{j+1}^{m+1} - 2V_j^{m+1} + V_{j-1}^{m+1}}{(\Delta x)^2} - (1-\theta)\frac{V_{j+1}^m - 2V_j^m + V_{j-1}^m}{(\Delta x)^2},$$

and so the consistency error (denoted T_j^m for parabolic and hyperbolic equations) is

$$\begin{aligned}T_j^m &= \mathcal{L}_h(u(x_j, t_m) - U_j^m) \\ &= \frac{u_j^{m+1} - u_j^m}{\Delta t} - \theta \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{(\Delta x)^2} - (1-\theta)\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2}\end{aligned}$$

for $j \in \mathbb{Z}$, and $m = 0, 1, \ldots, M-1$. which should look familiar.

Note that we used the notation T_j^m to be consistent with the notes. However, one could replace T_j^m with T_j^{m+1} instead. For some reason, the T_j^{m+1} convention is used in the solutions to the MMSC 2020 paper for the implicit Euler scheme ($\theta = 1$). The same convention is also used for the implicit scheme for an wave equation, as seen in the next section.

4 Wave equation

Let $\Omega = (-\infty, \infty) \times (0, T)$, where $T > 0$ is the final time and consider the wave equation:

$$\begin{aligned}\partial_{tt} u - c^2 \partial_{xx} u &= f \qquad \text{in } \Omega, \\ u|_{t=0} &= u_0 \qquad \text{on } \mathbb{R}, \\ \partial_t u|_{t=0} &= u_1 \qquad \text{on } \mathbb{R}.\end{aligned}$$

Note that we are using $\mathbb{R}$ for the spatial domain instead of a finite interval (a, b) for convenience. An implicit scheme for the wave equation is

$$\begin{aligned}\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{(\Delta t)^2} - c^2 \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} &= f(x_j, t_{m+1}) \qquad j \in \mathbb{Z},\ m = 1, 2, \ldots, M-1, \\ U_j^0 &= u_0(x_j) \qquad j \in \mathbb{Z}, \\ U_j^1 &= U_j^0 + \Delta t u_1(x_j) \qquad j \in \mathbb{Z},\end{aligned}$$

where $\Delta t = T/M$. Then, the definition of $\mathcal{L}_h$ depends on the time step. For $m = 1, 2, \ldots, M-1$, we have

$$\mathcal{L}_{h,m} V_j^m := \frac{V_j^{m+1} - 2V_j^m + V_j^{m-1}}{(\Delta t)^2} - c^2 \frac{V_{j+1}^{m+1} - 2V_j^{m+1} + V_{j-1}^{m+1}}{(\Delta x)^2}, \qquad j \in \mathbb{Z},$$

while for $m = 0$, we have

$$\mathcal{L}_{h,0} V_j^m := \frac{V_j^1 - V_j^0}{\Delta t} \qquad j \in \mathbb{Z}.$$

Then, the consistency error is

$$\begin{aligned} T_j^{m+1} &:= \mathcal{L}_{h,m}(u(x_j, t_m) - U_j^m) \\ &= \frac{u_j^{m+1} - 2u_j^m + u_j^{m-1}}{(\Delta t)^2} - c^2 \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{(\Delta x)^2} - f(x_j, t_{m+1}) \end{aligned}$$

for $j \in \mathbb{Z}$ and $m = 1, 2, \ldots, M-1$, and

$$T_j^1 := \mathcal{L}_{h,0}(u(x_j, t_1) - U_j^1) = \frac{u_j^1 - u_j^0}{\Delta t} - u_1(x_j) \qquad j \in \mathbb{Z}.$$

Note that here, we used a different convention than for the heat equation above.

An explicit scheme for the wave equation is

$$\begin{aligned} \frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{(\Delta t)^2} - c^2 \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} &= f(x_j, t_m) \qquad && j \in \mathbb{Z},\ m = 1, 2, \ldots, M-1, \\ U_j^0 &= u_0(x_j) && j \in \mathbb{Z}, \\ U_j^1 &= U_j^0 + \Delta t u_1(x_j) && j \in \mathbb{Z}, \end{aligned}$$

where $\Delta t = T/M$. Then, $\mathcal{L}_{h,m}$ is given by

$$\mathcal{L}_{h,m} V_j^m := \frac{V_j^{m+1} - 2V_j^m + V_j^{m-1}}{(\Delta t)^2} - c^2 \frac{V_{j+1}^m - 2V_j^m + V_{j-1}^m}{(\Delta x)^2}$$

for $j \in \mathbb{Z}$ and $m = 1, 2, \ldots, M-1$, and

$$\mathcal{L}_{h,0} V_j^1 := \frac{V_j^1 - V_j^0}{\Delta t} \qquad j \in \mathbb{Z}.$$

Then, the consistency error is

$$\begin{aligned} T_j^m &:= \mathcal{L}_{h,m}(u(x_j, t_m) - U_j^m) \\ &= \frac{u_j^{m+1} - 2u_j^m + u_j^{m-1}}{(\Delta t)^2} - c^2 \frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2} - f(x_j, t_m) \end{aligned}$$

for $j \in \mathbb{Z}$ and $m = 1, 2, \ldots, M-1$, and

$$T_j^0 := \mathcal{L}_{h,0}(u(x_j, t_1) - U_j^1) = \frac{u_j^1 - u_j^0}{\Delta t} - u_1(x_j) \qquad j \in \mathbb{Z}.$$

Note that here, we used the same convention as for the heat equation above.