

# Prelims Probability

## Sheet 4 — MT24

1. Let  $\mathbb{P}(X = k) = 1/n$  for  $k = 1, 2, \dots, n$ . Find the mean and variance of  $X$ .
2. Suppose that the discrete random variables  $X$  and  $Y$  have joint probability mass function given by

	$X$	-1	0	1
$Y$				
-1		$\frac{1}{27}$	$\frac{6}{27}$	$\frac{2}{27}$
0		$\frac{2}{27}$	$\frac{6}{27}$	$\frac{1}{27}$
1		$\frac{3}{27}$	$\frac{2}{27}$	$\frac{4}{27}$

Find the marginal distributions of  $X$  and  $Y$ . What is the covariance of  $X$  and  $Y$ ? Are  $X$  and  $Y$  independent?

3. Suppose that  $X$  and  $Y$  are independent Poisson random variables with parameters  $\lambda$  and  $\mu$  respectively. Find
  - (a) the joint probability mass function  $\mathbb{P}(X = k, Y = m)$ ;
  - (b)  $\mathbb{P}(X + Y = n)$  (what is this distribution?);
  - (c)  $\mathbb{P}(X = k | X + Y = n)$  (what is this distribution?);
  - (d)  $\mathbb{E}[X | X + Y = n]$ .
4. Let  $X$  and  $Y$  be independent random variables, both with Geometric( $p$ ) distribution.
  - (a) Find  $\mathbb{P}(X = k | X + Y = n + 1)$ , for  $k \in \{1, 2, \dots, n\}$ .
  - (b) Find the distribution of  $\min\{X, Y\}$ . [*Hint: consider  $\mathbb{P}(\min\{X, Y\} > k$ ), and see Question 3(a) of sheet 3.*]
5.
  - (a) A set of lecture notes has  $n$  pages. The number of typos on each page is a Poisson random variable with parameter  $\lambda$ , and is independent of the number of typos on all other pages. What is the expected number of pages with no typos?
  - (b) When reading the notes, you detect each typo with probability  $p$ , independently of detecting others. Let  $M$  denote the number of typos on a particular page and let  $D$  denote the number that you detect on that page. Write down  $\mathbb{P}(D = k | M = m)$ . Hence, for each  $k \geq 0$ , find  $\mathbb{P}(D = k)$ .

6. Let  $X$  and  $Y$  be discrete random variables. Show that the following two definitions of independence of  $X$  and  $Y$  are equivalent:

(i) For all  $x, y \in \mathbb{R}$ ,

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y);$$

(ii) For all  $A, B \subseteq \mathbb{R}$ ,

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B).$$

Show that if  $X$  and  $Y$  are independent, then also  $f(X)$  and  $g(Y)$  are independent for any functions  $f, g: \mathbb{R} \rightarrow \mathbb{R}$ .

7. Solve the following recurrence relations:

1.  $u_{n+1} = 3u_n + 2$  with  $u_0 = 0$ .

2.  $u_{n+1} = 2u_n + n$  with  $u_0 = 1$ .

3.  $u_{n+1} - 5u_n + 6u_{n-1} = 2$  with  $u_0 = u_1 = 1$ .

4.  $u_{n+1} - 3u_n + 2u_{n-1} = 1$  with  $u_0 = u_1 = 0$ .