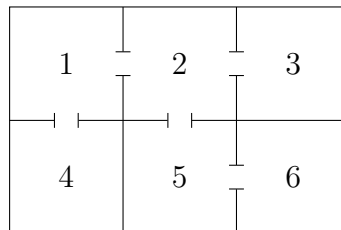


Prelims Probability

Sheet 5 — MT24

1. A bug jumps around the vertices of a triangle. At every jump, it moves from its current position to either of the other two vertices with probability $1/2$ each (independently of how it arrived at its current position). The bug starts at vertex 1. Let p_n be the probability that it is at vertex 1 after n jumps.
 - (a) Find the value of p_n for each n . [*Hint: find an appropriate first-order linear recurrence relation.*]
 - (b) What happens to p_n as $n \rightarrow \infty$?
2. The diagram below shows the floor plan of a house with six rooms: in room 1 is a mouse which will change rooms every minute, first moving at $t = 1$ and choosing a door to an adjoining room at random. In room 6 is a sleeping but hungry cat which will instantly wake if the mouse should enter. How long on average can we expect the mouse to survive?



3. (**Gambler's ruin, symmetric case.**) A gambler starts a game with a bankroll of $\mathcal{L}n$ where $n \in \{1, 2, \dots, M-1\}$. At each step of the game, he wins $\mathcal{L}1$ with probability $1/2$ and loses $\mathcal{L}1$ with probability $1/2$, independently for different steps. The game ends when the gambler's bankroll reaches $\mathcal{L}0$ or $\mathcal{L}M$.

In lectures we saw that the probability the gambler finishes with $\mathcal{L}M$ is n/M .

- (a) What is the expected amount of money that the gambler has at the end of the game?
- (b) Suppose we know that the gambler ends the game with $\mathcal{L}M$. What is the conditional probability that he won $\mathcal{L}1$ on the first step?
- (c) Let e_n be the expected length of the game. Find e_n for each n . For which n is e_n largest?

4. (a) Suppose that X has a geometric distribution with parameter p . Show that the probability generating function of X is

$$G_X(s) = \frac{ps}{1 - (1-p)s}, \quad \text{for } |s| < \frac{1}{1-p}.$$

- (b) Use this to calculate the mean and variance of X .

5. (a) A fair coin is tossed n times. Let r_n be the probability that the sequence of tosses never has a head followed by a head. Show that

$$r_n = \frac{1}{2}r_{n-1} + \frac{1}{4}r_{n-2}, \quad n \geq 2.$$

Find r_n using the conditions $r_0 = r_1 = 1$. Check that the value you get for r_2 is correct.

- (b) Let X be the number of coin tosses needed until you first get two heads in a row. (Note that $X \geq 2$.) Find the probability mass function of X .
- (c) Find the probability generating function of X . Use this to calculate the mean of X . (*You may wish to check that your answer agrees with what you got for Question 6 on Problem Sheet 3!*)
- (d) Let Y be the number of coin tosses needed until you first see a tail followed by a head. On any two particular coin tosses, the probability of seeing the pattern TH is $1/4$, the same as the probability of seeing the pattern HH. Therefore $\mathbb{P}(Y = 2) = \mathbb{P}(X = 2) = 1/4$. Find $\mathbb{P}(Y > n)$ for $n \geq 1$ and compare it to $\mathbb{P}(X > n)$. Is your answer surprising?

6. (*Optional. If you liked the coupon collector problem on Problem Sheet 3, you may enjoy this question too!*)

Consider a symmetric random walk on a cycle with N sites, labelled $0, 1, 2, \dots, N-1$. A particle starts at site 0, and at each step it jumps from its current site i to one of its two neighbours $i+1 \bmod N$ and $i-1 \bmod N$ with equal probability (independently of how it arrived at its current position).

- (a) Find the expected number of steps until every site has been visited. [*Hint: just after a new site has been visited, what does the set of visited sites look like? The value $e_1 = M-1$ from Question 3(c) may be useful!*]
- (b) For each $k = 1, \dots, N-1$, what is the probability that k is the last site to be visited? [*Hint: before visiting site k , the walk must visit either site $k-1$ or site $k+1$. What needs to happen from that point onwards?*]