

# Prelims Probability

## Sheet 6 — MT24

- If  $X$  is a constant random variable, say  $\mathbb{P}(X = a) = 1$  for some  $a \in \mathbb{N}$ , what is its probability generating function?
  - If  $Y$  has probability generating function  $G_Y(s)$ , and  $m, n$  are positive integers, what is the probability generating function of  $Z = mY + n$ ?
- Suppose that we perform a sequence of independent trials, each of which has probability  $p$  of success. Let  $Y$  be the number of trials up to and including the  $m$ th success, where  $m \geq 1$  is fixed. Explain why

$$\mathbb{P}(Y = k) = \binom{k-1}{m-1} p^m (1-p)^{k-m}, \quad k = m, m+1, \dots$$

(This is called the *negative binomial* distribution.)

- By expressing  $Y$  as a sum of  $m$  independent random variables, find its probability generating function.
- Let  $X_1, X_2, \dots$  be a sequence of independent and identically distributed non-negative integer valued random variables, and let  $N$  be a non-negative integer valued random variable which is independent of the sequence  $X_1, X_2, \dots$ .

Let  $Z = X_1 + \dots + X_N$  (where we take  $Z = 0$  if  $N = 0$ ).

- Show that

$$\mathbb{E}[Z] = \mathbb{E}[N]\mathbb{E}[X_1]$$

and

$$\text{var}(Z) = \text{var}(N)(\mathbb{E}[X_1])^2 + \mathbb{E}[N]\text{var}(X_1).$$

- If  $N \sim \text{Po}(\lambda)$  and  $X_1 \sim \text{Ber}(p)$ , find  $\text{var}(Z)$ .
  - [*Optional*] Suppose we remove the condition that  $N$  is independent of the sequence  $(X_i)$ . Is it still necessarily the case that  $\mathbb{E}[Z] = \mathbb{E}[N]\mathbb{E}[X_1]$ ? Find a proof or a counterexample.
- A random variable  $X$  has probability generating function  $G_X$ . Find a simple expression using  $G_X$  for the probability that  $X$  is even. [*Hint: consider the value of  $G_X(-1)$ . Possible extension: suggest a similar expression for the probability that  $X$  is divisible by 4 – be creative about what values of the generating function you might evaluate!*]

5. A population of cells is grown on a petri dish. Once a minute, each cell tries to reproduce by splitting in two. This is successful with probability  $1/4$ ; with probability  $1/12$ , the cell dies instead; and with the remaining probability  $2/3$ , nothing happens. Assume that different cells behave independently and that we begin with a single cell. What is the probability generating function  $G(s)$  of the number of cells on the dish after 1 minute? How about after 2 minutes? What is the probability that after 2 minutes the population has died out?
6. Consider a branching process in which each individual has 2 offspring with probability  $p$ , and 0 offspring with probability  $1 - p$ . Let  $X_n$  be the size of the  $n$ th generation, with  $X_0 = 1$ .
  - (a) Write down the mean  $\mu$  of the offspring distribution, and its probability generating function  $G(s)$ .
  - (b) Find the probability that the process eventually dies out. [*Recall that this probability is the smallest non-negative solution of the equation  $s = G(s)$ .*] Verify that the probability that the process survives for ever is positive if and only if  $\mu > 1$ .
  - (c) Let  $\beta_n = \mathbb{P}(X_n > 0)$ , the probability the process survives for at least  $n$  generations. Write down  $G(s)$  in the case  $p = 1/2$ . Deduce that in that case,

$$\beta_n = \beta_{n-1} - \beta_{n-1}^2/2,$$

and use induction to prove that, for all  $n$ ,

$$\frac{1}{n+1} \leq \beta_n \leq \frac{2}{n+2}.$$

- (d) [*For further exploration!*] In lectures we considered a simple random walk, which at each step goes up with probability  $p$  and down with probability  $1 - p$ . Suppose the walk starts from site 1. By taking limits in the gambler's ruin model, we showed that the probability that the walk ever hits site 0 equals 1 for  $p \leq 1/2$ , and  $(1-p)/p$  for  $p > 1/2$ .

Compare this probability to your answer in part (b). Can you find a link between the branching process and the random walk? [*Hint: if I take an individual in the branching process and replace it by its children (if any), what happens to the size of the population?*]