Prelims Mathematics 2024-25

September 25, 2024

Contents

1 Foreword

Synopses

The synopses give some additional detail and show how the material is split between the different lecture courses. They include details of recommended reading.

Practical Work

The requirement in the *Examination Regulations* to pursue an adequate course of practical work will be satisfied by following the Computational Mathematics course and submitting two Computational Mathematics projects. Details about submission of these projects will be given in the Computational Mathematics handbook.

2 Syllabus

The syllabus here is that referred to in the *Examination Regulations 2025* Special Regulations for the Preliminary Examination in Mathematics (`https://examregs.admin.ox.ac.uk/`) and has been approved by the Mathematics Teaching Committee for examination in Trinity Term 2025.

Examination Conventions can be found at: `http://www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/examination-conventions`

A The subject of the examination shall be mathematics and its applications. The syllabus and number of papers shall be prescribed by regulation from time to time by the Mathematical, Physical and Life Sciences Board.

B

1. Candidates shall take five written papers. The titles of the papers shall be: Mathematics I, Mathematics II, Mathematics III, Mathematics IV, Mathematics V.

2. In addition to the five papers in cl 1, a candidate must also offer a practical work assessment.

3. Candidates shall be deemed to have passed the examination if they have satisfied the Moderators in all five papers and the practical assessment at a single examination or passed all five papers and the practical assessment in accordance with the proviso of cl 4.

4. A candidate who fails to satisfy the Moderators in one or two of papers I-V may offer those papers on one subsequent occasion; a candidate who fails to satisfy the Moderators in three or more of papers I-V may offer all five papers on one subsequent occasion; a candidate who fails to satisfy the Moderators in the practical work assessment may also offer the assessment on one subsequent occasion.

5. The Moderators may award a distinction to candidates of special merit who have passed all five written papers and the practical work assessment at a single examination.

6. The syllabus for each paper shall be published by the Mathematical Institute on the Mathematical Institute's website by the beginning of the Michaelmas Full Term in the academic year of the examination, after consultation with the Mathematics Teaching Committee. Each paper will contain questions of a straight forward character.

7. The Chair of Mathematics, or a deputy, shall make available to the Moderators evidence showing the extent to which each candidate has pursued an adequate course of practical work. In assessing a candidate's performance in the examination the Moderators shall take this evidence into account. Deadlines for handing in practical work will be published in a handbook for candidates by the beginning of Michaelmas Full Term in the academic year of the examination.

Candidates are usually required to submit such practical work electronically; details shall be given in the handbook for the practical course. Any candidate who is unable for some reason to submit work electronically must apply to the Academic Administrator, Mathematical Institute, for permission to submit the work in paper form. Such applications must reach the Academic Administrator two weeks before the deadline for submitting the practical work.

8. The use of hand held pocket calculators is generally not permitted but certain kinds may

be permitted for some papers. Specifications of which papers and which types of calculator are permitted for those exceptional papers will be announced by the Moderators in the Hilary Term preceding the examination.

Mathematics I

The natural numbers and their ordering. Induction as a method of proof, including a proof of the binomial theorem with non-negative integral coefficients.

Sets. Examples including $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and intervals in $\mathbb{R}$. Inclusion, union, intersection, power set, ordered pairs and cartesian product of sets. Relations. Definition of an equivalence relation. Examples.

Functions: composition, restriction; injective (one-to-one), surjective (onto) and invertible functions; images and preimages.

Systems of linear equations. Matrices and the beginnings of matrix algebra. Use of matrices to describe systems of linear equations. Elementary Row Operations (EROs) on matrices. Reduction of matrices to echelon form. Application to the solution of systems of linear equations.

Inverse of a square matrix. Reduced row echelon (RRE) form and the use of EROs to compute inverses; computational efficiency of the method. Transpose of a matrix; orthogonal matrices.

Vector spaces: definition of a vector space over a field (such as $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$). Subspaces. Many explicit examples of vector spaces and subspaces.

Span of a set of vectors. Examples such as row space and column space of a matrix. Linear dependence and independence. Bases of vector spaces; examples. The Steinitz Exchange Lemma; dimension. Application to matrices: row space and column space, row rank and column rank. Coordinates associated with a basis of a vector space.

Use of EROs to find bases of subspaces. Sums and intersections of subspaces; the dimension formula. Direct sums of subspaces.

Linear transformations: definition and examples (including projections associated with direct-sum decompositions). Some algebra of linear transformations; inverses. Kernel and image, Rank-Nullity Theorem. Applications including algebraic characterisation of projections (as idempotent linear transformations).

Matrix of a linear transformation with respect to bases. Change of Bases Theorem. Applications including proof that row rank and column rank of a matrix are equal.

Bilinear forms; real inner product spaces; examples. Mention of complex inner product spaces. Cauchy–Schwarz inequality. Distance and angle. The importance of orthogonal matrices.

Introduction to determinant of a square matrix: existence and uniqueness. Proof of existence by induction. Proof of uniqueness by deriving explicit formula from the properties of the determinant. Permutation matrices. (No general discussion of permutations). Basic properties of determinant, relation to volume. Multiplicativity of the determinant, computation by row operations.

Determinants and linear transformations: definition of the determinant of a linear transformation, multiplicativity, invertibility and the determinant.

Eigenvectors and eigenvalues, the characteristic polynomial, trace. Eigenvectors for distinct eigenvalues are linearly independent. Discussion of diagonalisation. Examples. Eigenspaces, geometric and algebraic multiplicity of eigenvalues. Eigenspaces form a direct sum.

Gram-Schmidt procedure. Spectral theorem for real symmetric matrices. Quadratic forms and real symmetric matrices. Application of the spectral theorem to putting quadrics into normal form by orthogonal transformations and translations. Statement of classification of orthogonal transformations.

Axioms for a group and for an Abelian group. Examples including geometric symmetry groups, matrix groups (GL_n, SL_n, O_n, U_n), cyclic groups. Products of groups.

Permutations of a finite set under composition. Cycles and cycle notation. Order. Transpositions; every permutation may be expressed as a product of transpositions. The parity of a permutation is well-defined via determinants. Conjugacy in permutation groups.

Subgroups; examples. Intersections. The subgroup generated by a subset of a group. A subgroup of a cyclic group is cyclic. Connection with hcf and lcm. Bezout's Lemma.

Recap on equivalence relations including congruence mod n and conjugacy in a group. Proof that equivalence classes partition a set. Cosets and Lagrange's Theorem; examples. The order of an element. Fermat's Little Theorem.

Isomorphisms, examples. Groups of order 8 or less up to isomorphism (stated without proof). Homomorphisms of groups with motivating examples. Kernels. Images. Normal subgroups. Quotient groups; examples. First Isomorphism Theorem. Simple examples determining all homomorphisms between groups.

Group actions; examples. Definition of orbits and stabilizers. Transitivity. Orbits partition the set. Stabilizers are subgroups.

Orbit-stabilizer Theorem. Examples and applications including Cauchy's Theorem and to conjugacy classes.

Orbit-counting formula. Examples.

The representation $G \to \mathrm{Sym}(S)$ associated with an action of G on S. Cayley's Theorem. Symmetry groups of the tetrahedron and cube.

Mathematics II

Complex numbers and their arithmetic. The Argand diagram (complex plane). Modulus and argument of a complex number. Simple transformations of the complex plane. De Moivre's Theorem; roots of unity. Euler's theorem; polar form $re^{\mathrm{i}\theta}$ of a complex number. Polynomials and a statement of the Fundamental Theorem of Algebra.

Real numbers: arithmetic, ordering, suprema, infima; the real numbers as a complete ordered field. Definition of a countable set. The countability of the rational numbers. The reals are uncountable. The complex number system. The triangle inequality.

Sequences of real or complex numbers. Definition of a limit of a sequence of numbers. Limits and inequalities. The algebra of limits. Order notation: O, o.

Subsequences; a proof that every subsequence of a convergent sequence converges to the same limit; bounded monotone sequences converge. Bolzano–Weierstrass Theorem. Cauchy's convergence criterion.

Series of real or complex numbers. Convergence of series. Simple examples to include geometric progressions and some power series. Absolute convergence, Comparison Test, Ratio Test, Integral Test. Alternating Series Test.

Power series, radius of convergence. Examples to include definition of and relationships between exponential, trigonometric functions and hyperbolic functions.

Definition of the function limit. Definition of continuity of functions on subsets of $\mathbb{R}$ and $\mathbb{C}$ in terms of ε and δ. Continuity of real valued functions of several variables. The algebra of continuous functions; examples, including polynomials. Intermediate Value Theorem for continuous functions on intervals. Boundedness, maxima, minima and uniform continuity for continuous functions on closed intervals. Monotone functions on intervals and the Inverse Function Theorem.

Sequences and series of functions, uniform convergence. Weierstrass's M-test for uniformly convergent series of functions. Uniform limit of a sequence of continuous functions is continuous. Continuity of functions defined by power series.

Definition of the derivative of a function of a real variable. Algebra of derivatives, examples to include polynomials and inverse functions. The derivative of a function defined by a power series is given by the derived series (proof not examinable). Vanishing of the derivative at a local maximum or minimum. Rolle's Theorem, Mean Value Theorem, and Cauchy's (Generalized) Mean Value Theorem with applications: Constancy Theorem, monotone functions, exponential function and trigonometric functions. L'Hôpital's Formula. Taylor's Theorem with remainder in Lagrange's form; examples. The binomial expansion with arbitrary index.

Step functions, their integral, basic properties. Minorants and majorants of bounded functions on bounded intervals. Definition of Riemann integral. Elementary properties of Riemann integrals: positivity, linearity, subdivision of the interval.

The application of uniform continuity to show that continuous functions are Riemann integrable on closed bounded intervals; bounded continuous functions are Riemann integrable on bounded intervals.

The Mean Value Theorem for Integrals. The Fundamental theorem of Calculus; integration by parts and by substitution.

The interchange of integral and limit for a uniform limit of integrable functions on a bounded interval. Term-by-term integration and differentiation of a (real) power series (interchanging limit and derivative for a series of functions where the derivatives converge uniformly).

Mathematics III

General linear homogeneous ODEs: integrating factor for first order linear ODEs, second solution when one solution is known for second order linear ODEs. First and second order linear ODEs with constant coefficients. General solution of linear inhomogeneous ODE as particular solution plus solution of homogeneous equation. Simple examples of finding particular integrals by guesswork.

Introduction to partial derivatives. Second order derivatives and statement of condition for equality of mixed partial derivatives. Chain rule, change of variable, including planar polar coordinates. Solving some simple partial differential equations (e.g. $f_{xy} = 0$, $f_x = f_y$).

Parametric representation of curves, tangents. Arc length. Line integrals.

Jacobians with examples including plane polar coordinates. Some simple double integrals calculating area and also $\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\mathrm{d}A$.

Simple examples of surfaces, especially as level sets. Gradient vector; normal to surface; directional derivative; $\int_A^B \nabla\phi \cdot \mathrm{d}\mathbf{r} = \phi(B) - \phi(A)$.

Taylor's Theorem for a function of two variables (statement only). Critical points and classification using directional derivatives and Taylor's theorem. Informal (geometrical) treatment of Lagrange multipliers.

Sample space, events, probability measure. Permutations and combinations, sampling with or without replacement. Conditional probability, partitions of the sample space, law of total probability, Bayes' Theorem. Independence.

Discrete random variables, probability mass functions, examples: Bernoulli, binomial, Poisson, geometric. Expectation, expectation of a function of a discrete random variable, variance. Joint distributions of several discrete random variables. Marginal and conditional distributions. Independence. Conditional expectation, law of total probability for expectations. Expectations of functions of more than one discrete random variable, covariance, variance of a sum of dependent discrete random variables.

Solution of first and second order linear difference equations. Random walks (finite state space only).

Probability generating functions, use in calculating expectations. Examples including random sums and branching processes.

Continuous random variables, cumulative distribution functions, probability density functions, examples: uniform, exponential, gamma, normal. Expectation, expectation of a function of a continuous random variable, variance. Distribution of a function of a single continuous random variable. Joint probability density functions of several continuous random variables (rectangular regions only). Marginal distributions. Independence. Expectations of functions of jointly continuous random variables, covariance, variance of a sum of dependent jointly continuous random variables.

Random sample, sums of independent random variables. Markov's inequality, Chebyshev's inequality, Weak Law of Large Numbers.

Random samples, concept of a statistic and its distribution, sample mean as a measure of location and sample variance as a measure of spread.

Concept of likelihood; examples of likelihood for simple distributions. Estimation for a single unknown parameter by maximising likelihood. Examples drawn from: Bernoulli, binomial, geometric, Poisson, exponential (parametrized by mean), normal (mean only, variance known). Data to include simple surveys, opinion polls, archaeological studies, etc. Properties of estimators—unbiasedness, Mean Squared Error = (bias^2 + variance). Statement of Central Limit Theorem (excluding proof). Confidence intervals using CLT. Simple straight line fit, $Y_t = a + bx_t + \varepsilon_t$, with ε_t normal independent errors of zero mean and common known variance. Estimators for a, b by maximising likelihood using partial differentiation, unbiasedness and calculation of variance as linear sums of Y_t. (No confidence intervals). Examples (use scatter plots to show suitability of linear regression).

Linear regression with 2 regressors. Special case of quadratic regression $Y_t = a+bx_t+cx_t^2+\epsilon_t$. Model diagnostics and outlier detection. Residual plots. Heteroscedasticity. Outliers and studentized residuals. High-leverage points and leverage statistics.

Introduction to unsupervised learning with real world examples. Principal components analysis (PCA). Proof that PCs maximize directions of maximum variance and are orthogonal using Lagrange multipliers. PCA as eigendecomposition of covariance matrix. Eigenvalues as variances. Choosing number of PCs. The multivariate normal distribution pdf. Examples of PCA on multivariate normal data and clustered data.

Clustering techniques; K-means clustering. Minimization of within-cluster variance. K-means algorithm and proof that it will decrease objective function. Local versus global optima and use of random initializations. Hierarchical clustering techniques. Agglomerative clustering using complete, average and single linkage.

Mathematics IV

Euclidean geometry in two and three dimensions approached by vectors and coordinates. Vector addition and scalar multiplication. The scalar product. Equations of planes, lines and circles.

The vector product in three dimensions. Use of $\mathbf{a}, \mathbf{b}, \mathbf{a} \wedge \mathbf{b}$ as a basis. Scalar triple products and vector triple products, vector algebra.

Conics (normal form), focus and directrix. Degree two equations in two variables.

Orthogonal matrices and the maps they represent in $\mathbb{R}^2$. Orthogonal change of variables. Statement of Spectral Theorem and use in simple examples. Identifying conics not in normal form.

3×3 orthogonal matrices; SO(3) and rotations; conditions for being a reflection. Isometries of $\mathbb{R}^3$. Rotating frames in 2 and 3 dimensions. Angular velocity.

Parametrised surfaces, including spheres, cones; surfaces of revolution. Examples of finding shortest path on a surface. Surface isometries. Arc length and surface area; isometries and area.

Newton's laws and inertial frames. Dimensional analysis.

Forces: examples including gravity, fluid drag, electromagnetism. Energy and momentum.

Equilibria and the harmonic oscillator. Stability and instability via linearized equations, normal modes. Simple examples of equilibria in two variables via matrices.

Planar motion in polar coordinates. Conservative forces, central forces, angular momentum and torque. Constrained motion.

Newtonian gravitational potential, Kepler's laws and planetary motion.

Many particle systems, centre of mass motion, Galilean relativity.

Rigid bodies, the inertia tensor, and simple rigid body motion (with fixed axis of rotation). Newton's laws in rotating frames.

The Division Algorithm on Integers, Euclid's Algorithm including proof of termination with highest common factor. The solution of simple linear Diophantine equations. Examples.

Division and Euclid's algorithm for real polynomials. Examples.

Root finding for real polynomials. Fixed point iterations, examples. Convergence. Existence of fixed points and convergence of fixed point iterations by the contraction mapping theorem (using the mean value theorem).

Newton iteration. Quadratic convergence. Horner's Rule.

Mathematics V

Multiple integrals: Two dimensions. Informal definition and evaluation by repeated integration; example over a rectangle; properties. General domains. Change of variables. Examples.

Volume integrals: Jacobians for cylindrical and spherical polars, examples.

Recap on surface and line integrals. Flux integrals including solid angle. Work integrals and conservative fields.

Scalar and vector fields. Vector differential operators: divergence and curl; physical interpretation. Calculation. Identities.

Divergence theorem. Examples. Consequences: Green's first and second theorems. $\int_V \nabla\phi \mathrm{d}V = \int_{\partial V} \phi \mathrm{d}S$.

Uniqueness of solutions of Poisson's equation. Derivation of heat equation. Divergence theorem in plane.

Stokes's theorem. Examples. Consequences. The existence of potential for a conservative force.

Gauss' Flux Theorem. Examples. Equivalence with Poisson's equation.

Fourier series: Periodic, odd and even functions. Calculation of sine and cosine series. Simple applications concentrating on imparting familiarity with the calculation of Fourier coefficients and the use of Fourier series. The issue of convergence is discussed informally with examples. The link between convergence and smoothness is mentioned, together with its consequences for approximation purposes.

Partial differential equations: Introduction in descriptive mode on partial differential equations and how they arise. Derivation of

(i) the wave equation of a string,

(ii) the heat equation in one dimension (box argument only). Examples of solutions and their interpretation. D'Alembert's solution of the wave equation and applications. Characteristic diagrams (excluding reflection and transmission). Uniqueness of solutions of wave and heat equations.

PDEs with Boundary conditions. Solution by separation of variables. Use of Fourier series to solve the wave equation, Laplace's equation and the heat equation (all with two independent variables). (Laplace's equation in Cartesian and in plane polar coordinates). Applications.

3 Introduction to University Mathematics

3.1 Overview

The purpose of these introductory lectures is to establish some of the basic language and notation of university mathematics, and to introduce the elements of naïve set theory and the nature of formal proof.

3.2 Learning Outcomes

Students should:

(i) have the ability to describe, manipulate, and prove results about sets and functions using standard mathematical notation;

(ii) know and be able to use simple relations;

(iii) develop sound reasoning skills;

(iv) have the ability to follow and to construct simple proofs, including proofs by mathematical induction (including strong induction, minimal counterexample) and proofs by contradiction;

(v) learn how to write clear and rigorous mathematics.

3.3 Synopsis

The natural numbers and their ordering. Induction as a method of proof, including a proof of the binomial theorem with non-negative integral coefficients.

Sets. Examples including $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and intervals in $\mathbb{R}$. Inclusion, union, intersection, power set, ordered pairs and cartesian product of sets. Relations. Definition of an equivalence relation. Examples.

Functions: composition, restriction; injective (one-to-one), surjective (onto) and invertible functions; images and preimages.

Writing mathematics. The language of mathematical reasoning; quantifiers: "for all", "there exists". Formulation of mathematical statements with examples.

Proofs and refutations: standard techniques for constructing proofs; counter-examples. Example of proof by contradiction and more on proof by induction.

Problem-solving in mathematics: experimentation, conjecture, confirmation, followed by explaining the solution precisely.

3.4 Reading List

1) C. J. K. Batty, *How do undergraduates do Mathematics?*, (Mathematical Institute Study Guide, 1994)

2) K. Houston, *How to think like a mathematician*, (CUP, 2009)

3) L. Alcock, *How to study for a mathematics degree*, (OUP, 2012)

3.5 Further Reading

1) G. Pólya. *How to solve it: a new aspect of mathematical method*, (1945, New edition 2014 with a foreword by John Conway, Princeton University Press).

2) G. C. Smith, *Introductory Mathematics: Algebra and Analysis*, (Springer-Verlag, London, 1998), Chapters 1 and 2.

3) Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis*, (Wiley, New York, Fourth Edition, 2011), Chapter 1 and Appendices A and B.

4) C. Plumpton, E. Shipton, R. L. Perry, *Proof*, (MacMillan, London, 1984).

5) R. B. J. T. Allenby, *Numbers and Proofs*, (Butterworth-Heinemann, London, 1997).

6) R. A. Earl, *Bridging Material on Induction*, (Mathematics Department website).

4 Introduction to Complex Numbers

4.1 Overview

Students should not necessarily expect a tutorial to support this short course. Solutions to the problem sheet will be posted on Monday of Week 2 and students are asked to mark their own problems when no tutorial has been offered.

This course aims to give all students a common background in complex numbers.

4.2 Learning Outcomes

By the end of the course, students will be able to:

(i) manipulate complex numbers with confidence;

(ii) use the Argand diagram representation of complex numbers, including to solve problems involving the nth roots of unity;

(iii) know the polar representation form and be able to apply it in a range of problems.

4.3 Synopsis

Complex numbers and their arithmetic. The Argand diagram (complex plane). Modulus and argument of a complex number. Simple transformations of the complex plane. De Moivre's Theorem; roots of unity. Euler's theorem; polar form $re^{\mathrm{i}\theta}$ of a complex number. Polynomials and a statement of the Fundamental Theorem of Algebra.

4.4 Reading List

1) R. A. Earl, *Complex numbers* `https://www.maths.ox.ac.uk/study-here/undergraduate-study/bridging-gap` These notes have now been expanded as Chapter 1 of Towards Higher Mathematics (Cambridge University Press, 2017)

2) D. W. Jordan & P Smith, *Mathematical Techniques* (Oxford University Press, Oxford, 2002), Ch.6. Please note that e-book versions of many books in the reading lists can be found on SOLO `http://solo.bodleian.ox.ac.uk/primo-explore/search?vid=SOLO`Links to an external site. and ORLO `https://oxford.rl.talis.com/index.html`(Links to an external site.).

5 M1: Linear Algebra I

5.1 Overview

Linear algebra pervades and is fundamental to algebra, geometry, analysis, applied mathematics, statistics, and indeed most of mathematics. This course lays the foundations, concentrating mainly on vector spaces and matrices over the real and complex number systems. The course begins with examples in $\mathbb{R}^2$ and $\mathbb{R}^3$, and gradually becomes more abstract. The course also introduces the idea of an inner product, with which angle and distance can be introduced into a vector space.

5.2 Learning Outcomes

By the end of the course, students will be able to:

(i) use the definitions of a vector space, a subspace, linear dependence and independence, spanning sets and bases, both within the familiar setting of $\mathbb{R}^2$ and $\mathbb{R}^3$ and also for abstract vector spaces, and prove results using these definitions;

(ii) use matrices to solve systems of linear equations and to determine the number of solutions of such a system;

(iii) solve a range of problems relating to linear maps between vector spaces, thinking of linear maps abstractly or representing them using matrices as appropriate.

5.3 Synopsis

Systems of linear equations. Matrices and the beginnings of matrix algebra. Use of matrices to describe systems of linear equations. Elementary Row Operations (EROs) on matrices. Reduction of matrices to echelon form. Application to the solution of systems of linear equations.

Inverse of a square matrix. Reduced row echelon (RRE) form and the use of EROs to compute inverses; computational efficiency of the method. Transpose of a matrix; orthogonal matrices.

Vector spaces: definition of a vector space over a field (such as $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$). Subspaces. Many explicit examples of vector spaces and subspaces.

Span of a set of vectors. Examples such as row space and column space of a matrix. Linear dependence and independence. Bases of vector spaces; examples. The Steinitz Exchange Lemma; dimension. Application to matrices: row space and column space, row rank and column rank. Coordinates associated with a basis of a vector space.

Use of EROs to find bases of subspaces. Sums and intersections of subspaces; the dimension formula. Direct sums of subspaces.

Linear transformations: definition and examples (including projections associated with direct-sum decompositions). Some algebra of linear transformations; inverses. Kernel and image, Rank-Nullity Theorem. Applications including algebraic characterisation of projections (as idempotent linear transformations).

Matrix of a linear transformation with respect to bases. Change of Bases Theorem. Applications including proof that row rank and column rank of a matrix are equal.

Bilinear forms; real inner product spaces; examples. Mention of complex inner product spaces. Cauchy–Schwarz inequality. Distance and angle. The importance of orthogonal matrices.

5.4 Reading List

Basic Linear Algebra by T.S.Blyth and E.F.Robertson (2002, Springer) Guide to Linear Algebra by David Towers (1988, MacMillan) Linear Algebra – An Introductory Approach by Charles W Curtis (1984, Springer) Introduction to Linear Algebra by Gilbert Strang (2016, Wellesley-Cambridge)

6 M1: Linear Algebra II

6.1 Overview

To follow.

6.2 Learning Outcomes

Students will:

(i) understand the elementary theory of determinants;

(ii) understand the beginnings of the theory of eigenvectors and eigenvalues and appreciate the applications of diagonalizability.

(iii) understand the Spectral Theory for real symmetric matrices, and appreciate the geometric importance of an orthogonal change of variable.

6.3 Synopsis

Introduction to determinant of a square matrix: existence and uniqueness. Proof of existence by induction. Proof of uniqueness by deriving explicit formula from the properties of the determinant. Permutation matrices. (No general discussion of permutations). Basic properties of determinant, relation to volume. Multiplicativity of the determinant, computation by row operations.

Determinants and linear transformations: definition of the determinant of a linear transformation, multiplicativity, invertibility and the determinant.

Eigenvectors and eigenvalues, the characteristic polynomial, trace. Eigenvectors for distinct eigenvalues are linearly independent. Discussion of diagonalisation. Examples. Eigenspaces, geometric and algebraic multiplicity of eigenvalues. Distinct-eigenvalue eigenvectors are linearly independent.

Gram-Schmidt procedure. Spectral theorem for real symmetric matrices. Quadratic forms and real symmetric matrices. Application of the spectral theorem to putting quadrics into normal form by orthogonal transformations and translations.

6.4 Reading List

1. T. S. Blyth and E. F. Robertson, *Basic Linear Algebra* (Springer, London, 2nd edition 2002).
2. C. W. Curtis, *Linear Algebra – An Introductory Approach* (Springer, New York, 4th edition, reprinted 1994).
3. R. B. J. T. Allenby, *Linear Algebra* (Arnold, London, 1995).
4. D. A. Towers, *A Guide to Linear Algebra* (Macmillan, Basingstoke 1988).
5. S. Lang, *Linear Algebra* (Springer, London, Third Edition, 1987).
6. R. Earl, *Towards Higher Mathematics - A Companion* (Cambridge University Press, Cambridge, 2017)

7 M1: Groups and Group Actions

7.1 Overview

Abstract algebra evolved in the twentieth century out of nineteenth century discoveries in algebra, number theory and geometry. It is a highly developed example of the power of generalisation and axiomatisation in mathematics. The *group* is an important first example of an abstract, algebraic structure and groups permeate much of mathematics particularly where there is an aspect of symmetry involved. Moving on from examples and the theory of groups, we will also see how groups *act* on sets (e.g. permutations on sets, matrix groups on vectors) and apply these results to several geometric examples and more widely.

7.2 Learning Outcomes

Students will get familiarised with the axiomatic approach to group theory and learn how to argue formally and abstractly. They will be ale to apply the First Isomorphism Theorem and work with many examples of groups and group actions from various parts of mathematics. With the help of the Counting Lemma (also called Burnside's Lemma) they will be able to solve a variety of otherwise intractable counting problems and thus learn to appreciate the power of groups.

7.3 Synopsis

HT (8 lectures)

Axioms for a group and for an Abelian group. Examples including geometric symmetry groups, matrix groups (GL_n, SL_n, O_n, U_n), cyclic groups. Products of groups.

Permutations of a finite set under composition. Cycles and cycle notation. Order. Transpositions; every permutation may be expressed as a product of transpositions. The parity of a permutation is well-defined via determinants. Conjugacy in permutation groups.

Subgroups; examples. Intersections. The subgroup generated by a subset of a group. A subgroup of a cyclic group is cyclic. Connection with hcf and lcm. Bezout's Lemma.

Recap on equivalence relations including congruence mod n and conjugacy in a group. Proof that equivalence classes partition a set. Cosets and Lagrange's Theorem; examples. The order of an element. Fermat's Little Theorem.

TT (8 Lectures)

Isomorphisms, examples. Groups of order 8 or less up to isomorphism (stated without proof). Homomorphisms of groups with motivating examples. Kernels. Images. Normal subgroups. Quotient groups; examples. First Isomorphism Theorem. Simple examples determining all homomorphisms between groups.

Group actions; examples. Definition of orbits and stabilizers. Transitivity. Orbits partition the set. Stabilizers are subgroups.

Orbit-stabilizer Theorem. Examples and applications including Cauchy's Theorem and to conjugacy classes.

Orbit-counting formula. Examples.

The representation $G \to \mathrm{Sym}(S)$ associated with an action of G on S. Cayley's Theorem. Symmetry groups of the tetrahedron and cube.

7.4 Reading List

1) M. A. Armstrong *Groups and Symmetry* (Springer, 1997)

7.5 Further Reading

1) R. B. J. T. Allenby, *Rings, Fields and Groups* (Second revised edition, Elsevier, 1991)

2) Peter J. Cameron, *Introduction to Algebra* (Second edition, Oxford University Press, 2007).

3) John B. Fraleigh, *A First Course in Abstract Algebra* (Seventh edition, Pearson, 2013).

4) W. Keith Nicholson, *Introduction to Abstract Algebra* (Fourth edition, John Wiley, 2012).

5) Joseph J. Rotman, *A First Course in Abstract Algebra* (Third edition, Pearson, 2005).

6) Joseph Gallian, *Contemporary Abstract Algebra* (8th international edition, Brooks/Cole, 2012).

7) Nathan Carter, *Visual Group Theory* (MAA Problem Book Series, 2009).

8 M2: Analysis I - Sequences and Series

8.1 Overview

In these lectures we study the real and complex numbers, and study their properties, particularly completeness (roughly speaking, the idea that there are no 'gaps' - unlike in the rational numbers, for example). We go on to define and study limits of sequences, convergence of series, and power series.

8.2 Learning Outcomes

By the end of the course, students will be able to:

- prove results within an axiomatic framework;
- define and prove basic results about countable and uncountable sets, including key examples;
- define what it means for a sequence or series to converge;
- prove results using the completeness axiom for $\mathbb{R}$ and using Cauchy's criterion for the convergence of real and complex sequences and series, and explain how completeness and Cauchy's criterion are related;
- analyse the convergence (or otherwise) of a variety of well known sequences and series, and use this to conjecture the behaviour of unfamiliar sequences and series;
- apply standard techniques to determine whether a sequence converges, and whether a series converges;
- define the elementary functions using power series, and use these definitions to deduce basic properties of these functions.

8.3 Synopsis

Real numbers: arithmetic, ordering, suprema, infima; the real numbers as a complete ordered field. Definition of a countable set. The countability of the rational numbers. The reals are uncountable. The complex number system. The triangle inequality.

Sequences of real or complex numbers. Definition of a limit of a sequence of numbers. Limits and inequalities. The algebra of limits. Order notation: O, o.

Subsequences; a proof that every subsequence of a convergent sequence converges to the same limit; bounded monotone sequences converge. Bolzano–Weierstrass Theorem. Cauchy's convergence criterion.

Series of real or complex numbers. Convergence of series. Simple examples to include geometric progressions and some power series. Absolute convergence, Comparison Test, Ratio Test, Integral Test. Alternating Series Test.

Power series, radius of convergence. Examples to include definition of and relationships between exponential, trigonometric functions and hyperbolic functions.

8.4 Reading List

1. Alcock, L. (2014) How to think about analysis. First edition. Oxford: Oxford University Press.

2. Bartle, R. G. and Sherbert, D. R. (2000a) Introduction to real analysis. 3rd ed. New York: Wiley.

3. Bartle, R. G. and Sherbert, D. R. (2011) Introduction to real analysis. 4th ed. New York: Wiley.

4. Bressoud, D. M. and Mathematical Association of America (2007) A radical approach to real analysis. [s.l.]: Mathematical Association of America.

5. Bryant, V. (1990) Yet another introduction to analysis. Cambridge: Cambridge University Press.

6. Burkill, J. C. (1978) A first course in mathematical analysis. 1st paperback ed. Cambridge: Cambridge University Press.

7. Burn, R. P. (2015) Numbers and functions: steps into analysis. Third edition. Cambridge: Cambridge University Press.

8. Hart, F. M. (1988) Guide to analysis. Basingstoke: Macmillan Education.

9. Hart, F. M. (2001) Guide to analysis. 2nd ed. Basingstoke: Palgrave.

10. Howie, J. M. (2001) Real analysis. London: Springer.

11. Smith, G. (1998) Introductory mathematics: algebra and analysis. London: Springer.

12. Spivak, M. (1980) Calculus. 2nd ed. Berkeley, CA: Publish or Perish.

13. Spivak, M. (1994) Calculus. 3rd ed. Houston: Publish or Perish.

14. Thomson, B., Bruckner, J. B. and Bruckner, A. M. (2008) Elementary real analysis. 2nd ed. California: Createspace Publishing.

9 M2: Analysis II - Continuity and Differentiability

9.1 Overview

In this term's lectures, we study continuity of functions of a real or complex variable, and differentiability of functions of a real variable.

9.2 Learning Outcomes

At the end of the course students will be able to apply limiting properties to describe and prove continuity and differentiability conditions for real and complex functions. They will be able to prove important theorems, such as the Intermediate Value Theorem, Rolle's Theorem and Mean Value Theorem, and will continue the study of power series and their convergence.

9.3 Synopsis

Definition of the function limit. Definition of continuity of functions on subsets of $\mathbb{R}$ and $\mathbb{C}$ in terms of ε and δ. The algebra of continuous functions; examples, including polynomials. Intermediate Value Theorem for continuous functions on intervals. Boundedness, maxima, minima and uniform continuity for continuous functions on closed intervals. Monotone functions on intervals and the Continuous Inverse Function Theorem.

Sequences and series of functions, uniform convergence. Weierstrass's M-test for uniformly convergent series of functions. Uniform limit of a sequence of continuous functions is continuous. Continuity of functions defined by power series.

Definition of the derivative of a function of a real variable. Algebra of derivatives, examples to include polynomials and inverse functions. The derivative of a function defined by a power series is given by the derived series (proof not examinable). Vanishing of the derivative at a local maximum or minimum. Rolle's Theorem, Mean Value Theorem, and Cauchy's (Generalized) Mean Value Theorem with applications: Constancy Theorem, monotone functions, exponential function and trigonometric functions. L'Hôpital's Formula. Taylor's Theorem with remainder in Lagrange's form; examples. The binomial expansion with arbitrary index.

9.4 Reading List

1) Lecture Notes for this course.

2) W. Rudin, *Principles of Mathematical Analysis* (McGraw-Hill, Third Edition), Chapters 4, 5, 7.

3) T. M. Apostol, *Mathematical Analysis* (Addison-Wesley Pub. Company), Chapters 4 and 5.

4) M. Spivak, *Calculus* (Cambridge University Press; 3 edition), Sections 5 to 12.

9.5 Further Reading

1) M. Giaquinta and G. Modica, *Mathematical Analysis: Functions of One Variable: v. 1* (Birkhäuser).

2) V. A. Zorich, *Mathematical Analysis I* (2nd Edition, Universitext, Springer).

10 M2: Analysis III - Integration

10.1 Overview

In these lectures we define Riemann integration and study its properties, including a proof of the Fundamental Theorem of Calculus. This gives us the tools to justify term-by-term differentiation of power series and deduce the elementary properties of the trigonometric functions.

10.2 Learning Outcomes

At the end of the course students will be familiar with the construction of an integral from fundamental principles, including important theorems. They will know when it is possible to integrate or differentiate term-by-term and be able to apply this to, for example, trigonometric series.

10.3 Synopsis

Step functions, their integral, basic properties. Minorants and majorants of bounded functions on bounded intervals. Definition of Riemann integral. Elementary properties of Riemann integrals: positivity, linearity, subdivision of the interval.

The application of uniform continuity to show that continuous functions are Riemann integrable on closed bounded intervals; bounded continuous functions are Riemann integrable on bounded intervals.

The Mean Value Theorem for Integrals. The Fundamental theorem of Calculus; integration by parts and by substitution.

The interchange of integral and limit for a uniform limit of integrable functions on a bounded interval. Term-by-term integration and differentiation of a (real) power series (interchanging limit and derivative for a series of functions where the derivatives converge uniformly).

10.4 Reading List

Lecture notes will be provided

10.5 Further Reading

1) W. Rudin, *Principles of Mathematical Analysis* (McGraw-Hill, Third Edition, 1976).

This is a more advanced text containing more material than is in the course, including the Stieltjes integral.

11 M3: Introductory Calculus

11.1 Overview

These lectures are designed to give students a gentle introduction to applied mathematics in their first term at Oxford, allowing time for both students and tutors to work on developing and polishing the skills necessary for the course. It will have an 'A-level' feel to it, helping in the transition from school to university. The emphasis will be on developing skills and familiarity with ideas using straightforward examples.

11.2 Learning Outcomes

At the end of the course, students will be able to solve a range of ordinary differential equations (ODEs). They will also be able to evaluate partial derivatives and use them in a variety of applications.

11.3 Synopsis

General linear homogeneous ODEs: integrating factor for first order linear ODEs, second solution when one solution is known for second order linear ODEs. First and second order linear ODEs with constant coefficients. General solution of linear inhomogeneous ODE as particular solution plus solution of homogeneous equation. Simple examples of finding particular integrals by guesswork. [4]

Introduction to partial derivatives. Second order derivatives and statement of condition for equality of mixed partial derivatives. Chain rule, change of variable, including planar polar coordinates. Solving some simple partial differential equations (e.g. $f_{xy} = 0$, $f_x = f_y$). [3.5]

Parametric representation of curves, tangents. Arc length. Line integrals. [1]

Jacobians with examples including plane polar coordinates. Some simple double integrals calculating area and also $\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\mathrm{d}A$. [2]

Simple examples of surfaces, especially as level sets. Gradient vector; normal to surface; directional derivative; $\int_A^B \nabla\phi \cdot \mathrm{d}\mathbf{r} = \phi(B) - \phi(A)$.[2]

Taylor's Theorem for a function of two variables (statement only). Critical points and classification using directional derivatives and Taylor's theorem. Informal (geometrical) treatment of Lagrange multipliers.[3.5]

11.4 Reading List

1) M. L. Boas, *Mathematical Methods in the Physical Sciences* (Wiley, 3rd Edition, 2005).

2) D. W. Jordan & P. Smith, *Mathematical Techniques* (Oxford University Press, 3rd Edition, 2003).

3) E. Kreyszig, *Advanced Engineering Mathematics* (Wiley, 10th Edition, 2011).

4) K. A. Stroud, *Advanced Engineering Mathematics* (Palgrave Macmillan, 5th Edition, 2011).

12 M3: Probability

12.1 Overview

An understanding of random phenomena is becoming increasingly important in today's world within social and political sciences, finance, life sciences and many other fields. The aim of this introduction to probability is to develop the concept of chance in a mathematical framework. Random variables are introduced, with examples involving most of the common distributions.

12.2 Learning Outcomes

Students should have a knowledge and understanding of basic probability concepts, including conditional probability. They should know what is meant by a random variable, and have met the common distributions. They should understand the concepts of expectation and variance of a random variable. A key concept is that of independence which will be introduced for events and random variables.

12.3 Synopsis

Sample space, events, probability measure. Permutations and combinations, sampling with or without replacement. Conditional probability, partitions of the sample space, law of total probability, Bayes' Theorem. Independence.

Discrete random variables, probability mass functions, examples: Bernoulli, binomial, Poisson, geometric. Expectation, expectation of a function of a discrete random variable, variance. Joint distributions of several discrete random variables. Marginal and conditional distributions. Independence. Conditional expectation, law of total probability for expectations. Expectations of functions of more than one discrete random variable, covariance, variance of a sum of dependent discrete random variables.

Solution of first and second order linear difference equations. Random walks (finite state space only).

Probability generating functions, use in calculating expectations. Examples including random sums and branching processes.

Continuous random variables, cumulative distribution functions, probability density functions, examples: uniform, exponential, gamma, normal. Expectation, expectation of a function of a continuous random variable, variance. Distribution of a function of a single continuous random variable. Joint probability density functions of several continuous random variables (rectangular regions only). Marginal distributions. Independence. Expectations of functions of jointly continuous random variables, covariance, variance of a sum of dependent jointly continuous random variables.

Random sample, sums of independent random variables. Markov's inequality, Chebyshev's inequality, Weak Law of Large Numbers.

12.4 Reading List

1) G. R. Grimmett and D. J. A. Welsh, *Probability: An Introduction* (Oxford University Press, 1986), Chapters 1–4, 5.1–5.4, 5.6, 6.1, 6.2, 6.3 (parts of), 7.1–7.3, 10.4.

2) J. Pitman, *Probability* (Springer-Verlag, 1993).

3) S. Ross, *A First Course In Probability* (Prentice-Hall, 1994).

4) D. Stirzaker, *Elementary Probability* (Cambridge University Press, 1994), Chapters 1–4, 5.1–5.6, 6.1–6.3, 7.1, 7.2, 7.4, 8.1, 8.3, 8.5 (excluding the joint generating function).

13 M3: Statistics and Data Analysis

13.1 Overview

The course introduces the concept of likelihood for a probabilistic model and its use in estimating unknown model parameters. Models covered will include linear regression with one or two regressors. In many examples confidence intervals may be found by using the Central Limit Theorem (statement only). Model checking and outlier detection are core concepts that are broadly relevant across many aspects of mathematical modelling and will be explored here in the context of regression with one or two regressors. Regression models are an example of *supervised learning*, however a large part of statistics and data analysis can be classified as *unsupervised learning*, i.e. finding structure in data sets, e.g. data from financial markets, medical imaging, retail, population genetics, social networks. Techniques for finding structure in datasets are relevant to many parts of applied maths, specifically this course will cover principal components analysis and clustering techniques.

13.2 Learning Outcomes

Students should have an understanding of likelihood, the use of maximum likelihood to find estimators, and some properties of the resulting estimators. They should have an understanding of confidence intervals and their construction using the Central Limit Theorem. They should have an understanding of linear regression with one or two regressors, and of finding structure in data sets using principal components and some clustering techniques.

13.3 Synopsis

Random samples, concept of a statistic and its distribution, sample mean as a measure of location and sample variance as a measure of spread.

Concept of likelihood; examples of likelihood for simple distributions. Estimation for a single unknown parameter by maximising likelihood. Examples drawn from: Bernoulli, binomial, geometric, Poisson, exponential (parametrized by mean), normal (mean only, variance known). Data to include simple surveys, opinion polls, archaeological studies, etc. Properties of estimators—unbiasedness, Mean Squared Error = (bias^2 + variance). Statement of Central Limit Theorem (excluding proof). Confidence intervals using CLT. Simple straight line fit, $Y_t = a + bx_t + \varepsilon_t$, with ε_t normal independent errors of zero mean and common known variance. Estimators for a, b by maximising likelihood using partial differentiation, unbiasedness and calculation of variance as linear sums of Y_t. (No confidence intervals). Examples (use scatter plots to show suitability of linear regression). [8]

Linear regression with 2 regressors. Special case of quadratic regression $Y_t = a+bx_t+cx_t^2+\epsilon_t$. Model diagnostics and outlier detection. Residual plots. Heteroscedasticity. Outliers and studentized residuals. High-leverage points and leverage statistics. [2.5]

Introduction to unsupervised learning with real world examples. Principal components analysis (PCA). Proof that principal components (PCs) maximize directions of maximum variance and are orthogonal using Lagrange multipliers. PCA as eigendecomposition of covariance matrix. Eigenvalues as variances. Choosing number of PCs. The multivariate normal distribution pdf. Examples of PCA on multivariate normal data and clustered data. [3]

Clustering techniques; K-means clustering. Minimization of within-cluster variance. K-means algorithm and proof that it will decrease objective function. Local versus global optima and use of random initializations. Hierarchical clustering techniques. Agglomerative clustering using complete, average and single linkage [2.5]

13.4 Reading List

1) F. Daly, D. J. Hand, M. C. Jones, A. D. Lunn, K. J. McConway, *Elements of Statistics* (Addison Wesley, 1995). Chapters 1–5 give background including plots and summary statistics, Chapter 6 and parts of Chapter 7 are directly relevant.

2) G. James, D. Witten, T. Hastie, R. Tibshirani *An Introduction to Statistical Learning* (with Applications in R) (Springer 2013) - Chapter 3 on linear regression problems, Chapter 10 on unsupervised learning. This book is freely available online.

13.5 Further Reading

1) J. A. Rice, *Mathematical Statistics and Data Analysis* (Wadsworth and Brooks Cole, 1988).

2) S. Rogers and M. Girolami, *A First Course in Machine Learning* (CRC Press 2011)

14 M4: Geometry

14.1 Overview

The course is an introduction to some elementary ideas in the geometry of Euclidean space through vectors. One focus of the course is the use of coordinates and an appreciation of the invariance of geometry under an orthogonal change of variables. This leads to a deeper study of orthogonal matrices, of rotating frames, and into related coordinate systems. Another focus is on the construction and geometric properties of lines, planes, and surfaces.

14.2 Learning Outcomes

Students will learn how to encode a geometric scenario into vector equations and meet the vector algebra needed to manipulate such equations. Students will meet the benefits of choosing sensible co-ordinate systems and appreciate what geometry is invariant of such choices. Students will gain introductory insight into the geometry of surfaces and lines.

14.3 Synopsis

Euclidean geometry in two and three dimensions approached by vectors and coordinates. Vector addition and scalar multiplication. The scalar product. Equations of planes, lines and circles. [3]

The vector product in three dimensions. Use of $\mathbf{a}, \mathbf{b}, \mathbf{a} \wedge \mathbf{b}$ as a basis. Scalar triple products and vector triple products, vector algebra, intersection of lines and planes. [2]

Conics (normal form), focus, directrix, and properties. Degree two equations in two variables. [2]

Isometries. Orthogonal matrices and the maps they represent in $\mathbb{R}^2$. Orthonormal bases in $\mathbb{R}^3$. Orthogonal change of coordinates. Statement of Spectral Theorem and use in simple examples. [2]

3×3 orthogonal matrices; SO(3) and rotations; conditions for being a reflection. Isometries of $\mathbb{R}^n$. Rotating frames in 2 and 3 dimensions. Angular velocity. [2]

Parametrised surfaces, including spheres, cones; surfaces of revolution. Examples of finding shortest path on a surface. Surface isometries. Arc length and surface area; isometries and area. [4]

14.4 Reading List

- J. Roe, *Elementary Geometry,* Oxford Science Publications (1992), Chapters 1, 2.2, 3.4, 4, 5.3, 7.1–7.3, 8.1–8.3, 12.1.
- R. Earl. *Towards Higher Mathematics: A Companion*, Cambridge University Press (2017) Chapters 3.1, 3.2, 3.7, 3.10, 4.2, 4.3

15 M4: Dynamics

15.1 Overview

The subject of dynamics is about how things change with time. A major theme is the modelling of a physical system by differential equations, and one of the highlights involves using the law of gravitation to account for the motion of planets.

15.2 Learning Outcomes

Students will be familiar with the laws of motion, including circular and planetary motion. They will know how forces are used and be introduced to stability in a physical system.

15.3 Synopsis

Newton's laws and inertial frames. Dimensional analysis. [1.5]

Forces: examples including gravity, fluid drag, electromagnetism. Energy and momentum. [2.5]

Equilibria and the harmonic oscillator. Stability and instability via linearized equations, normal modes. Simple examples of equilibria in two variables via matrices. [2]

Planar motion in polar coordinates. Conservative forces, central forces, angular momentum and torque. Constrained motion. [3]

Newtonian gravitational potential, Kepler's laws and planetary motion. [2.5]

Many particle systems, centre of mass motion, Galilean relativity. [1.5]

Rigid bodies, the inertia tensor, and simple rigid body motion (with fixed axis of rotation). Newton's laws in rotating frames. [3]

15.4 Reading List

1) David Acheson, *From Calculus to Chaos: an Introduction to Dynamics* (Oxford University Press, 1997), Chapters 1, 5, 6, 10.

2) M. Lunn, *A First Course in Mechanics* (Oxford University Press, 1991), Chapters 1–3, 5, 6, 7 (up to 7.3).

3) M. W. McCall, *Classical Mechanics: A Modern Introduction* (Wiley, 2001), Chapters 1–4, 7.

16 M5: Multivariable Calculus

16.1 Overview

In these lectures, students will be introduced to multi-dimensional vector calculus. They will be shown how to evaluate volume, surface and line integrals in three dimensions and how they are related via the Divergence Theorem and Stokes' Theorem - these are in essence higher dimensional versions of the Fundamental Theorem of Calculus.

16.2 Learning Outcomes

Students will be able to perform calculations involving div, grad and curl, including appreciating their meanings physically and proving important identities. They will further have a geometric appreciation of three-dimensional space sufficient to calculate standard and non-standard line, surface and volume integrals. In later integral theorems they will see deep relationships involving the differential operators.

16.3 Synopsis

Multiple integrals: Two dimensions. Informal definition and evaluation by repeated integration; example over a rectangle; properties. General domains. Change of variables. Examples. [2]

Volume integrals: Jacobians for cylindrical and spherical polars, examples. [1.5]

Recap on surface and line integrals. Flux integrals including solid angle. Work integrals and conservative fields. [2]

Scalar and vector fields. Vector differential operators: divergence and curl; physical interpretation. Calculation. Identities. [2.5]

Divergence theorem. Proof for convex regions (non-examinable). Examples. Consequences: Green's first and second theorems. $\int_V \nabla\phi \mathrm{d}V = \int_{\partial V} \phi \mathrm{d}S$.

Uniqueness of solutions of Poisson's equation. Derivation of heat equation. Divergence theorem in plane. [4]

Stokes's theorem. Examples. Consequences. The existence of potential for a conservative force. [2]

Gauss' Flux Theorem. Examples. Equivalence with Poisson's equation. [2]

16.4 Reading List

D.W. Jordan & Peter Smith Mathematical Techniques: an introduction for engineering, physical and mathematical sciences (2008, OUP) Erwin Kreyszig, Herbert Kreyszig, E.J.Norminton Advanced Engineering Mathematics (2011, Wiley) .

17 M5: Fourier Series and PDE's

17.1 Overview

The course begins by introducing students to Fourier series, concentrating on their practical application rather than proofs of convergence. Students will then be shown how the heat equation, the wave equation and Laplace's equation arise in physical models. They will learn basic techniques for solving each of these equations in several independent variables, and will be introduced to elementary uniqueness theorems.

17.2 Learning Outcomes

Students will be familiar with Fourier series and their applications and be notionally aware of their convergence. Students will know how to derive the heat, wave and Laplace's equations in several independent variables and to solve them. They will begin the study of uniqueness of solution of these important PDEs.

17.3 Synopsis

Fourier series: Periodic, odd and even functions. Calculation of sine and cosine series. Simple applications concentrating on imparting familiarity with the calculation of Fourier coefficients and the use of Fourier series. The issue of convergence is discussed informally with examples. The link between convergence and smoothness is mentioned, together with its consequences for approximation purposes.

Partial differential equations: Introduction in descriptive mode on partial differential equations and how they arise. Derivation of

(i) the wave equation of a string,

(ii) the heat equation in one dimension (box argument only). Examples of solutions and their interpretation. D'Alembert's solution of the wave equation and applications. Characteristic diagrams (excluding reflection and transmission). Uniqueness of solutions of wave and heat equations.

PDEs with Boundary conditions. Solution by separation of variables. Use of Fourier series to solve the wave equation, Laplace's equation and the heat equation (all with two independent variables). (Laplace's equation in Cartesian and in plane polar coordinates). Applications.

17.4 Reading List

1) D. W. Jordan and P. Smith, *Mathematical Techniques* (Oxford University Press, 4th Edition, 2003)

2) E. Kreyszig, *Advanced Engineering Mathematics* (Wiley, 10th Edition, 1999)

3) G. F. Carrier and C. E. Pearson, *Partial Differential Equations — Theory and Technique* (Academic Press, 1988)

18 Computational Mathematics

18.1 Overview

Many mathematicians use general-purpose mathematical software which includes tools for symbolic and numerical computation and other features such as plotting, visualization and data analysis. Such software is used for solving linear and nonlinear equations, graphing the results, as a tool for exploring mathematical concepts, and as a technique for verifying the correctness of calculations done "by hand". In this introductory course, students will explore these ideas in the Python programming language.

Python: Python is the most popular programming language in the world, due to its simplicity, versatility, parsimony, power, and flexibility. Python is widely used in science and engineering, quantitative finance, web programming, computer gaming, and data science. Python code is executed by a Python interpreter; the Python interpreter is open-source software that may be freely installed on almost any computer. In this course we will employ the Visual Studio Code environment to provide a user-friendly code development and debugging experience. Students will install Python and Visual Studio Code on their own individual computers, or access this software on college computers. For the latter they should consult the computing support at their own college.

Teaching and Assessment: The course relies heavily on self-teaching through practical exercises. A manual for the course and examples to be worked through will be provided.

You are expected to go through the lecture notes and problem sheets at a steady pace, and you will be timetabled for sessions every fortnight where you will have access to help and advice from demonstrators. Clear instructions will be provided on how to install the necessary software; please install it in advance of these sessions.

Your work in this course will be taken into account as part of the Preliminary Examinations. For further information, see the section on examinations in the Undergraduate Handbook.

18.2 Learning Outcomes

Students will be introduced to the Python programming language, with a heavy emphasis on the use of programming to investigate and solve mathematical problems, both pure and applied. Students will develop confidence and expertise in a tool which can be used in the later years of the mathematics course and in their subsequent careers.

18.3 Synopsis

Using Python on the command line and working with ".py" files;

Plotting in two and three dimensions;

Works with lists, dictionaries, matrices and linear algebra;

Symbolic computing using sympy;

Logic, flow control and iteration;

Solving problems in algebra, calculus, and applied mathematics.

18.4 Reading List

1) *Computational Mathematics: Michaelmas Term Students' Guide*, available from the course website. Printed copies may be requested.

2) C. Hill. *Learning scientific programming with Python.* Cambridge University Press, second edition, 2020.

3) H. P. Langtangen. *A Primer on Scientific Programming with Python.* Springer Berlin Heidelberg, 2016.

4) J. VanderPlas. *A Whirlwind Tour of Python.* O'Reilly Media, Inc., 2016.