Prelims Dynamics

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1 Newtonian Mechanics

1.1 Space and Time

In Newtonian mechanics *Space* is Euclidean $\mathbb{R}^3$.

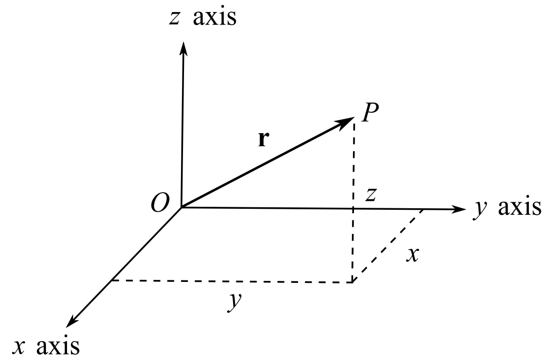


Figure 1: The position vector $\mathbf{r} = (x, y, z)$ of a point P, as measured in a reference frame $\mathcal{S}$.

Definition A *reference frame* $\mathcal{S}$ is specified by a choice of origin O, together with a set of orthogonal (right handed) 3D Cartesian coordinate axes at O.

A point P may be specified in this reference frame by its position vector $\mathbf{r} = \overrightarrow{OP}$.

The distance between two points with respective position vectors $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ is

$$\left((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right)^{1/2}.$$

In Newtonian mechanics *Time* is the same for all observers in all reference frames and can be measured by a clock.

In Newtonian physics it is further assumed that

- distances measured in two distinct reference frames are the same

Hence the mapping between two distinct reference frames is an isometry, though the reference frames may be such that the isometry changes with time.

From Prelims Geometry An isometry is a combination of a translation and an orthogonal transformation, and the latter can be represented by an orthogonal matrix $\mathcal{R}$. Preserving the right handedness of the reference frame entails $\det \mathcal{R} = 1$ and hence the orthogonal transformation is a rotation.

Definition A *point particle* is an idealised object, modelled as being located at position $\mathbf{r}(t)$ at time t relative to a reference frame.

Note that $\mathbf{r}(t)$ is a curve in 3D space, parameterised by time, t, and the *trajectory* of the particle.

Definition Some other basic definitions

$$\begin{aligned}
\text{Velocity} = \mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{\mathbf{r}} = \left(\frac{\mathrm{d}x}{\mathrm{d}t},\ \frac{\mathrm{d}y}{\mathrm{d}t},\ \frac{\mathrm{d}z}{\mathrm{d}t}\right) = (\dot{x},\ \dot{y},\ \dot{z})\,,\\
\text{Speed} = v = |\mathbf{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2},\\
\text{Acceleration} = \mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \ddot{\mathbf{r}} = \left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2},\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2},\ \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\right) = (\ddot{x},\ \ddot{y},\ \ddot{z})\,.
\end{aligned}$$

Example (Motion with constant acceleration).

- Suppose a particle has a constant acceleration $\mathbf{q}$ and starts at time $t = 0$ at the origin, with initial velocity $\mathbf{u}$. Find its trajectory.
- Simplify the expression for the trajectory if $\mathbf{q}$ and $\mathbf{u}$ are parallel.

Solution

We have $\ddot{\mathbf{r}} = \mathbf{q}$, constant.

Integrating, and using $\dot{\mathbf{r}}(0) = \mathbf{u}$, we have

$$\dot{\mathbf{r}} = \mathbf{q}t + \mathbf{u}.$$

Integrating again and using $\mathbf{r}(0) = \mathbf{0}$ yields

$$\mathbf{r} = \frac{1}{2}\mathbf{q}t^2 + \mathbf{u}t.$$

If $\mathbf{u}$ and $\mathbf{q}$ are parallel, WLOG let $\mathbf{q} = q\mathbf{k}$, $\mathbf{u} = u\mathbf{k}$ where $\mathbf{k}$ is the unit vector in the increasing z direction (appropriately chosen). Then

$$\mathbf{r}(t) = \left(\frac{1}{2}qt^2 + ut\right)\mathbf{k} = \left(0, 0, \frac{1}{2}qt^2 + ut\right).$$

1.2 Newton's Laws

Definition A point particle has a (*inertial*) *mass* $m > 0$. Its *linear momentum*, often abbreviated to *momentum*, is

$$\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}.$$

Aside As we will see from Newton's laws below we have that, loosely, inertial mass measures how difficult it is to accelerate the particle.

Definition *Newton's first law* states that

- In an *inertial reference frame* a particle moves with constant momentum, unless acted on by a non-zero total external force.

Not all reference frames are inertial.

On a stationary train a particle will rest on the floor, with gravity balanced by the normal reaction force from the floor, so that the total external force is zero. This is an inertial reference frame.

The same particle in a braking train will, to observers in the train, move forward and yet is subject to essentially zero external force if the floor has minimal friction.

Thus, an origin and axes fixed in the train do not constitute an inertial frame though they can mapped to the inertial frame with origin and axes fixed in the train station by a time dependent translation.

Aside Strictly, a reference frame at rest relative to the earth's surface is actually not an inertial frame. The Earth moves around the Sun and rotates on its own axis. The gravitational force due to the Sun is balanced by the acceleration required to remain in orbit around the sun and thus has no impact on whether or not the Earth's surface is an inertial frame. The Earth's rotation on its axis can be important though, for most purposes, approximating a reference frame at rest relative to the earth's surface as inertial has excellent accuracy. However with long range projectiles that are airborne for a long time, for example, an inertial reference frame is not legitimate. We return to this later.

Definition *Newton's second law* states that

- In an inertial reference frame the dynamics of a point particle is such that the rate of change of linear momentum is equal to the net force acting on the particle:

$$\mathbf{F} = \dot{\mathbf{p}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}).$$

Given constant particle mass, this reduces to

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}.$$

Finally, with more than one particle we have

Definition *Newton's third law* states that

- If particle 1 exerts a force $\mathbf{F} = \mathbf{F}_{21}$ on particle 2, then particle 2 also exerts a force $\mathbf{F}_{12} = -\mathbf{F}$ on particle 1, so that

$$\mathbf{F}_{12} = -\mathbf{F}_{21}.$$

This is often summarised as "every action has an equal and opposite reaction."

1.3 Galilean transformations

Inertial reference frames are not unique. Suppose we have an inertial frame $\mathcal{S}$, with respect to which positions are specified by a vector $\mathbf{r} = (x, y, z)$ from the origin O. Consider the following transformations to a different frame $\mathcal{S}'$, with positions specified by $\mathbf{r}'$:

$$\begin{cases} \text{spatial translations, } \mathbf{r}' = \mathbf{r} - \mathbf{x}, \text{ where } \mathbf{x} \text{ is a constant vector,} \\ \text{constant rotations, } \mathbf{r}' = \mathcal{R}\,\mathbf{r}, \text{ where } \mathcal{R} \text{ is a constant } 3\times 3 \text{ rotation matrix,} \\ \text{Galilean boosts, } \mathbf{r}' = \mathbf{r} - \mathbf{u}\,t, \text{ where } \mathbf{u} \text{ is a constant velocity.} \end{cases}$$

The first and second transformations simply translate the origin by a fixed distance, and rotate the axes by a fixed rotation, respectively. The final transformation has the origins O, O' moving at a fixed relative velocity $\mathbf{u}$.

If $\mathbf{r}(t)$ is the trajectory of a *free particle* – by definition, no forces act on it – in the frame $\mathcal{S}$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \mathbf{0}.$$

For each $\mathbf{r}'(t)$ above one has

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}' = \mathbf{0},$$

and hence the particle also moves with constant velocity in the new frame $\mathcal{S}'$.

Any combination of the above transformations thus maps an inertial frame to another inertial frame, generating the *Galilean transformation group.*

The insight of Galileo was that physics is *invariant* under Galilean transformations: the laws of motion are the same in any inertial frame. This is known as *Galileo's principle of relativity.*

1.4 Dimensions

With square bracket notation to denote the dimension of a variable, the fundamental dimensions in mechanics are:

$$[\text{length}] = \mathrm{L}, \quad [\text{time}] = \mathrm{T}, \quad [\text{mass}] = \mathrm{M}.$$

Aside Electric Charge, Q, is also required for electromagnetism.

Dimensions of other quantities may then be derived from these. For example, the dimensions of force, $\mathbf{F}$, are given by

$$[\mathbf{F}] = [m\mathbf{a}] = [m\ddot{\mathbf{r}}] = \mathrm{MLT}^{-2}.$$

In particular 1 kg m s^{-2} defines the unit of force, the *Newton.*

It is useful to note:

- A given dimension may be measured in a number of different standard *units.* For example, length may be measured in centimetres, or metres, or kilometers, etc.
- We may only add two quantities if they have the same dimensions and the units must match before adding.
- Functions that add terms of different powers, e.g.

  $$e^x = 1 + x + x^2/2 + \ldots...$$

  or sin, cos, tan etc., must act on dimensionless variables so that x above has dimensions of $\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0$.
- A convenient and powerful check on any equation you write down is whether the dimensions on both sides are the same.

1.4.1 Dimensional Analysis

A knowledge of the dimensions of the parameters a problem depends on can sometimes be used to construct scaling laws without needing to solve any differential equations.

Example (Maximum height for constant acceleration): For the previous example of a particle moving along the z axis, starting at the origin at time $t = 0$ with velocity $\mathbf{u} = u\mathbf{k}$, and constant acceleration $q\mathbf{k}$, we showed

$$\mathbf{r}(t) = (\frac{1}{2}qt^2 + ut)\,\mathbf{k}.$$

Suppose that $u > 0$ but the constant acceleration $q = -g < 0$ is negative. Find the maximum height, z_{max} attained by the particle.

Solution It will reach a maximum height $z_{\max}$ at time $t_{\max}$, when $\dot{\mathbf{r}}(t_{\max}) = \mathbf{0}$:

$$\mathbf{0} \quad = \quad \dot{\mathbf{r}}(t_{\max}) \quad = \quad (-g\,t_{\max} + u)\,\mathbf{k} \qquad \Longrightarrow \qquad t_{\max} \quad = \quad \frac{u}{g}\ . \tag{1.1}$$

Hence $t_{\max} \quad = \quad u/g$, and thus .

$$z_{\max} \quad = \quad -\frac{1}{2}g\,t_{\max}^2 + u\,t_{\max} \quad = \quad \frac{u^2}{2g}\ . \tag{1.2}$$

However note that the answer must be in terms of u, $q = -g$ as these are the only parameters in problem statement.

We have $[u] = \mathrm{L\,T^{-1}}$, $[g] = \mathrm{L\,T^{-2}}$. The only way to obtain a quantity with dimensions of L is

$$\left[\frac{u^2}{g}\right] \quad = \quad \frac{\mathrm{L^2\,T^{-2}}}{\mathrm{L\,T^{-2}}} \quad = \quad \mathrm{L}\ . \tag{1.3}$$

Hence $z_{\max}$ must be a dimensionless number times u^2/g.

2 Forces and dynamics: a first look

In this section we introduce a number of different forces, and solve the differential equation given by Newton's second law to find the particle trajectory $\mathbf{r}(t)$.

Note that if forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ all act on a particle, the force $\mathbf{F}$ appearing in Newton's second law is the total force given by the *vector sum*

$$\mathbf{F} \;=\; \sum_{i=1}^{n} \mathbf{F}_i \; . \tag{2.1}$$

2.1 Examples of forces

2.1.1 Gravity

1. The gravitational force on a particle near the Earth's surface is $\mathbf{F} = m_g \mathbf{g}$, where m_g is the gravitational mass of the particle, and $\mathbf{g}$ is the gravitational acceleration, a vector pointing downward with magnitude $g = |\mathbf{g}| = 9.81\mathrm{m\ s}^{-2}$.

   In an inertial frame fixed on the Earth's surface, with z pointing up, Newton's second law becomes
   $$m_g \mathbf{g} = -m_g g \mathbf{k} = \mathbf{F} = m_I \ddot{\mathbf{r}} = m_I \ddot{z} \mathbf{k},$$
   where $m_I = m$ is the inertial mass.

   It is an experimental fact that $m_I = m_G$, as demonstrated famously by Galileo throwing items off the tower of Pisa, and confirmed to over 10 significant figures more generally. Henceforth we assume $m_I = m_g = m$.

   Hence Newton's laws give $\ddot{r} = -g\mathbf{k}$, which we considered in the previous constant acceleration example.

2. The gravitational force on the Earth due to the Sun is
   $$\mathbf{F} = -\frac{G m_1 m_2}{r^2} \mathbf{e}_r \tag{2.2}$$
   where m_1 and m_2 are the masses of the Sun and Earth respectively, $\mathbf{r}$ is the position vector of the Earth relative to the centre of the Sun, with magnitude r and in the direction of the unit vector $\mathbf{e}_r$. G is the universal gravitational constant ($G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$).

Example Suppose that a small projectile is thrown with velocity $\mathbf{V}$ at an angle α to the horizontal, from a height h above the ground. Find the curve traced out by the trajectory of the projectile, and its horizontal range.

We choose the origin O at ground level, and a unit vector $\mathbf{k}$ pointing vertically, and $\mathbf{i}$ horizontally along the ground. The only force acting is gravity, with $\mathbf{F} = -mg\,\mathbf{k}$, so that Newton's second law gives the equation of motion

$$m\ddot{\mathbf{r}} \;=\; -mg\,\mathbf{k} \; . \tag{2.3}$$

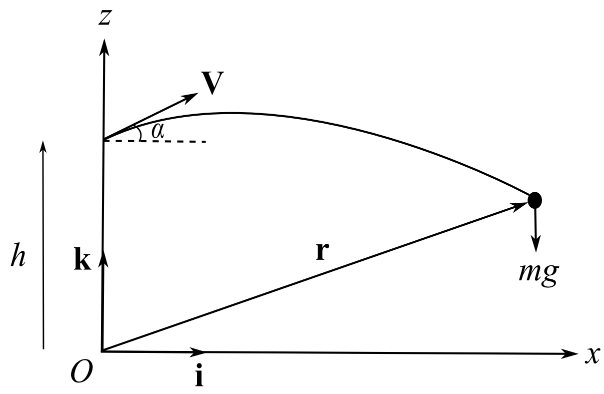


Figure 2: Throwing a projectile.

The initial conditions are

$$\text{At time } t=0: \qquad \mathbf{r}(0) \;=\; h\,\mathbf{k}\;, \qquad \dot{\mathbf{r}}(0) \;=\; \mathbf{V} \;=\; V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{k}\;. \tag{2.4}$$

Integrating (2.3) twice and using (2.4) we find the solution

$$\mathbf{r}(t) \;=\; -\frac{1}{2}g\,t^2\,\mathbf{k} + t\mathbf{V} + h\mathbf{k} = -\frac{1}{2}g\,t^2\,\mathbf{k} + tV\cos\alpha\,\mathbf{i} + tV\sin\alpha\,\mathbf{k} + h\,\mathbf{k}\;. \tag{2.5}$$

This is the trajectory of the projectile. We can find the curve that this traces out in the (x, z) plane by eliminating time t. Writing $\mathbf{r} = x\,\mathbf{i} + z\,\mathbf{k}$, reading off the components of (2.5) gives

$$x(t) \;=\; tV\cos\alpha\;, \qquad z(t) \;=\; -\frac{1}{2}g\,t^2 + tV\sin\alpha + h\;. \tag{2.6}$$

Using the first equation we may solve for t in terms of x, and then substitute into the second equation, giving the parabola

$$z \;=\; -\frac{g}{2V^2}x^2\sec^2\alpha + x\tan\alpha + h\;. \tag{2.7}$$

The projectile hits the ground when $z = \mathbf{r}\cdot\mathbf{k} = 0$. From (2.7) this gives a *quadratic* equation for the horizontal range x, with solution

$$x \;=\; \frac{V^2\cos\alpha}{g}\left[\sin\alpha + \sqrt{\sin^2\alpha + 2gh/V^2}\right]\;. \tag{2.8}$$

2.1.2 Normal Reaction Force and Friction

When a particle rests on a table, it experiences a force $m\mathbf{g}$ due to gravity. This is balanced exactly by a normal reaction force, often denoted $\mathbf{N}$, and ultimately of electrostatic origin.

If the particle slides across the table, or is acted on by a force tangential to the table top, friction can be generated which tends to act to oppose this motion and force.

2.1.3 Fluid drag

A particle moving through a fluid (such as air or water) experiences a drag force. Usually this is taken to act in the direction $-\mathbf{v}$, where $\mathbf{v}$ is the particle velocity.

For example, Stokes' law for a small sphere moving through a viscous liquid says the drag force is

$$\mathbf{F} = -6\pi\mu R\mathbf{v},$$

where μ is the dynamic viscosity of the liquid, R is the radius of the sphere.

A quadratic drag often holds for streamlined shapes such as aerofoils, whence

$$\mathbf{F} \;=\; -D\,|\dot{\mathbf{r}}|\,\dot{\mathbf{r}}\ , \qquad (2.9)$$

where the constant $D > 0$ and depends on the geometry of the aerofoil and the fluid density.

Example (Linear drag): Consider a particle falling under gravity with a linear drag force, $\mathbf{F} = -b\mathbf{v}$, with $b > 0$. The particle is released from rest at time $t = 0$. Determine its trajectory and terminal velocity.

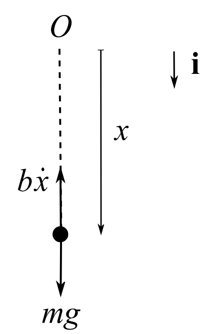


Figure 3: A particle falling under gravity with a linear drag.

Solution We choose an inertial frame with origin O, at the starting location of the particle and (unconventionally) take $\mathbf{i}$ to be a unit vector in the downwards direction, so that $\mathbf{r}(t) = x(t)\mathbf{i}$.

The force due to gravity is $mg\,\mathbf{i}$ and the drag force is $-b\,\dot{\mathbf{r}} = -b\dot{x}\mathbf{i}$.

Newton's second law gives the equation of motion

$$m\ddot{x}\,\mathbf{i} \;=\; mg\,\mathbf{i} - b\,\dot{x}\,\mathbf{i}\ . \qquad (2.10)$$

We hence deduce the one-dimensional equation

$$\ddot{x} \,+ \frac{b}{m}\,\dot{x} = g, \qquad (2.11)$$

with initial conditions $x(0) = \dot{x}(0) = 0$.

There are many ways to proceed. For instance with $q = \dot{x}$ we have $\dot{q} + (b/m)q = g$, which has a general solution of a multiple, A, of the homogeneous solution and a particular integral:

$$q \;=\; \frac{mg}{b} + A\mathrm{e}^{-bt/m}. \qquad (2.12)$$

Imposing the initial condition $q(0) = \dot{x}(0) = 0$ fixes A and we have

$$\dot{x} = q = \frac{mg}{b}\left(1 - \mathrm{e}^{-bt/m}\right). \tag{2.13}$$

Integrating again, with the initial condition $x(0) = 0$ yields

$$x(t) \;=\; \frac{mgt}{b} + \frac{m^2}{b^2}g\left(\mathrm{e}^{-bt/m} - 1\right). \tag{2.14}$$

$$x(t) \;=\; \frac{m^2 g}{b^2}\left(\mathrm{e}^{-\frac{b}{m}t} - 1\right) + \frac{mg}{b}t\,, \tag{2.15}$$

giving the trajectory. The *terminal velocity* is given by

$$\lim_{t\to\infty} \dot{x}(t) = \frac{mg}{b}.$$

2.1.4 Spring force

A spring is fixed at one end and attached to a particle at the other. The particle will experience a force, $\mathbf{F}$, directed along the line of the spring with a magnitude that depends on the extension of the spring from its equilibrium length.

Hooke's linear law states that the force is proportional to the extension, so that

$$\mathbf{F} = -k(x - l)\mathbf{t},$$

where $\mathbf{t}$ is the unit vector along the spring pointing towards the particle, x is the length of the spring, l is the equilibrium length, and k is the spring constant.

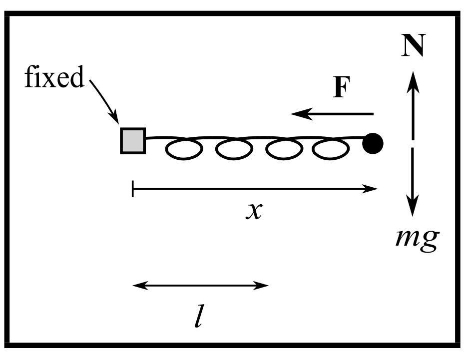


Figure 4: The forces acting on a particle attached to a spring on a table top. The normal reaction force, $\mathbf{N}$ and gravity, $m\mathbf{g}$ cancel so there is zero total force perpendicular to the x-axis.

Example Suppose a particle of mass m is attached to a spring that possesses a spring constant k, a length l, and is fixed by its opposing end to the origin, while aligned along the x-axis of a table top with no friction. Initially the particle is at location $x(0) = l$, with speed $\dot{x}(0) = u$ in the positive x direction. Determine its trajectory.

Solution Resolving Newton's second law along the x axis gives the equation of motion

$$m\ddot{x} \quad = \quad -k(x-l)\ , \tag{2.16}$$

and $x(0) = l$, $\dot{x}(0) = u$. With $x = l + q$ we have

$$\ddot{q} = -\omega^2 q, \quad \omega^2 = \frac{k}{m} > 0. \tag{2.17}$$

Thus $q = A\sin(\omega t + \psi)$, where A is the *amplitude* and ψ the *phase* of the ensuing oscillation. Imposing the initial conditions gives $A\omega = u$, $\psi = 0$ and hence

$$x(t) = l + \frac{u}{\omega}\sin\omega t.$$

2.1.5 Charged particle in electric and magnetic fields

For a charged particle moving in an electric field the force on the particle is the Lorentz force, given by

$$\mathbf{F} = e\mathbf{E} + e\mathbf{v} \wedge \mathbf{B},$$

where e is the charge on the particle, $\mathbf{E}$ is the electric field, $\mathbf{v}$ is the velocity of the particle, and $\mathbf{B}$ is the magnetic induction.

Example (Charged particle moving in a constant magnetic field): Ignoring gravity, determine the trajectory of a particle of charge q moving in *constant* magnetic field $\mathbf{B}$.

Solution Without loss, we take the initial location of the particle at time $t = 0$ to be the origin. We also denote its initial velocity by $\mathbf{V}$. From Newton's second law the equations of motion are:

$$m\ddot{\underline{r}} = q\dot{\underline{r}} \wedge \underline{B}, \quad \underline{r}(0) = \underline{0}, \quad \dot{\underline{r}}(0) = \underline{V}.$$

Integrating and using the initial conditions,

$$m\dot{\underline{r}} = q\underline{r} \wedge \underline{B} + m\underline{V}$$

WLOG $\underline{B} = (0, 0, B)$, $\underline{V} = (V_1, 0, V_3)$, $B > 0$ so that

$$m\dot{x} = qBy + mV_1$$
$$m\dot{y} = -qBx$$
$$m\dot{z} = mV_3$$

Hence $z = V_3 t$.

$$m\ddot{x} = qB\dot{y} = -q^2B^2/m \, x \quad \therefore \quad \ddot{x} = -\frac{q^2B^2}{m^2} x$$

Let $\omega = qB/m$. Then $\ddot{x} = -\omega^2 x$

Hence $x = A\sin(\omega t)$ noting $x(0) = 0$

Thus $x = \frac{V_1}{\omega} \sin(\omega t)$ using $\dot{x}(0) = V_1$

$$\therefore \; y(t) = \frac{m}{qB}\dot{x} - \frac{mV_1}{qB} = \frac{mV_1}{qB}\left(\cos(\omega t) - 1\right)$$
$$= \frac{V_1}{\omega}\left(\cos(\omega t) - 1\right)$$

$$\therefore \; \underline{r}(t) = \left(\frac{V_1}{\omega}\sin(\omega t), \; \frac{V_1}{\omega}\left(\cos(\omega t) - 1\right), \; V_3 t\right),$$

$$\omega = qB/m$$

3 Motion in one dimension

3D problems often reduce to lower dimensions and thus we first consider Newton's laws in more detail for one dimensional point particle mechanics.

3.1 Energy

Definition The *kinetic energy* of a point particle is

$$T = \frac{1}{2}m\dot{\mathbf{r}}^2.$$

Energy is measured in *Joules* J, with 1 J = 1 $\mathrm{kg\,m^2\,s^{-2}}$.

Consider a particle of constant mass, m, restricted to move along the x-axis, subject to a force $\mathbf{F} = F(x)\mathbf{i}$, as exemplified by the spring example of Section 2.1.4. First we define the potential energy:

Definition With x_0 an arbitrary fixed location, the particle's *potential energy* is

$$V(x) = -\int_{x_0}^{x} F(s)\,\mathrm{d}s.$$

Note that $V(x)$ is defined only up to an additive constant and that $\mathrm{d}V/\mathrm{d}x = -F(x)$.

3.1.1 Conservation of energy

Resolving Newton's second law in the x-direction for the above point particle gives the second order ODE

$$m\ddot{x} \quad = \quad F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}(x), \tag{3.1}$$

where $x = x(t)$ is a function of time. Multiplying both sides by $\dot{x}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2\right) = m\dot{x}\ddot{x} = -\dot{x}\frac{\mathrm{d}V}{\mathrm{d}x}(x) = -\frac{\mathrm{d}V(x(t))}{\mathrm{d}t},$$

with the first and last equalities from the product and chain rules of differentiation respectively. Hence

$$\frac{1}{2}m\dot{x}^2 + V(x) = E, \text{ constant }. \tag{3.2}$$

This equation describes CONSERVATION OF ENERGY:

$$\boxed{\text{Kinetic Energy, } T + \text{Potential Energy, } V = \text{Constant, } E\ .}$$

3.1.2 Work

For a small increment in the particle location, δx, the *work done* by the Force is $F(x)\delta x$. Hence more generally

Definition The *work done* W by the force in moving the particle from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\,\mathrm{d}x \ . \tag{3.3}$$

Suppose the particle starts at position x_1 at time t_1, and finishes at x_2 at time t_2. Then combining the definition of work done with $\mathrm{d}V/\mathrm{d}x = -F(x)$ immediately gives

$$W = -V(x_2(t_2)) + V(x_1(t_1)).$$

As $T + V$ is constant the change in V is minus the change in T for the particle trajectory and hence we have:

The Work-Energy Relation The work done by the force is the change in kinetic energy:

$$W = T(x_2(t_2)) - T(x_1(t_1)) \ . \tag{3.4}$$

3.1.3 Examples

Examples of the potential associated with forces, with suitable choices for the additive constant, include:

1. For gravity, $F(x) = -mg$, a choice of potential is $V(x) = mgx$.
2. For Hooke's law $F(x) = -k(x - l)$ a choice of potential is $V(x) = \frac{1}{2}k(x - l)^2$.
3. If the force, F, also depends on t or $\dot{x}$, as occurs with fluid drag and friction, it will often not be possible to write the integral of F over the trajectory as a function of x only. Then the conservation rules above will not hold in general. For example with friction, the system will lose energy as the particle moves (unless the heat energy generated by the friction is also accounted for). However the Lorentz force $\mathbf{F} = e\mathbf{E} + e\mathbf{v} \wedge \mathbf{B}$ is an exception.

Example (Maximum height under gravity, again): Consider once more a particle moving vertically under gravity, which at time $t = 0$ starts at height $z = 0$ with velocity $\dot{z} = u > 0$ upwards. What is the maximum height of the particle?

The potential is $V(z) = mgz$. The conserved energy E may be calculated from the initial conditions, which gives $E = T(0) = \frac{1}{2}mu^2$. Thus conservation of energy gives

$$\frac{1}{2}m\dot{z}^2 + mgz = \frac{1}{2}mu^2 \ . \tag{3.5}$$

The maximum height occurs when $\dot{z} = 0$, which immediately gives

$$z_{\max} = \frac{u^2}{2g} \ . \tag{3.6}$$

3.2 Motion in a general potential

Rearranging the equation for conservation of energy, (3.2), gives us

$$\dot{x}^2 \;=\; \frac{2}{m}(E - V(x)) \; . \tag{3.7}$$

This is a first order ODE, which we can in principle solve as

$$t \;=\; \pm \int \frac{\mathrm{d}x}{\left(\frac{2}{m}(E - V(x)\right)^{1/2}} \; . \tag{3.8}$$

for t as a function of x. We then invert to find $x(t)$.

This is often of limited utility in practice; apart from in very simple problems, we often cannot determine the integral nor invert.

Example (Quadratic potential – the simple harmonic oscillator): With $V(x) = kx^2/2$, which gives Hooke's law after a constant translation of x, we have Newton's second law reduces to

$$\ddot{x} + \omega^2 x \;=\; 0 \; . \tag{3.9}$$

with $\omega^2 = k/m$. This is the equation of motion for a simple harmonic oscillator.

We have already seen and solved this ODE, via Eqn. (2.17). Here we use Eqn. (3.8) which gives

$$t \;=\; \pm \int \frac{\mathrm{d}x}{\omega\sqrt{\frac{2E}{m\omega^2} - x^2}} \; . \tag{3.10}$$

We may solve this by making the substitution

$$x \;=\; \sqrt{\frac{2E}{m\omega^2}}\cos\theta \; , \tag{3.11}$$

which gives

$$t \;=\; \mp \int \frac{1}{\omega}\mathrm{d}\theta \qquad \Longrightarrow \qquad t - t_0 \;=\; \mp\frac{1}{\omega}\cos^{-1}\left(\frac{x}{\sqrt{2E/m\omega^2}}\right) \; . \tag{3.12}$$

Here t_0 is an integration constant. The solution is hence simple harmonic motion

$$x(t) \;=\; \sqrt{\frac{2E}{m\omega^2}}\cos\left[\omega(t - t_0)\right] \; . \tag{3.13}$$

Notice that in this case it is *much easier* to solve the second order equation of motion, than to integrate the first order conservation of energy equation.

Example Consider a particle moving in the *general* potential $V(x)$ shown in Figure 5. We can deduce *qualitative* aspects motion, using only conservation of energy.

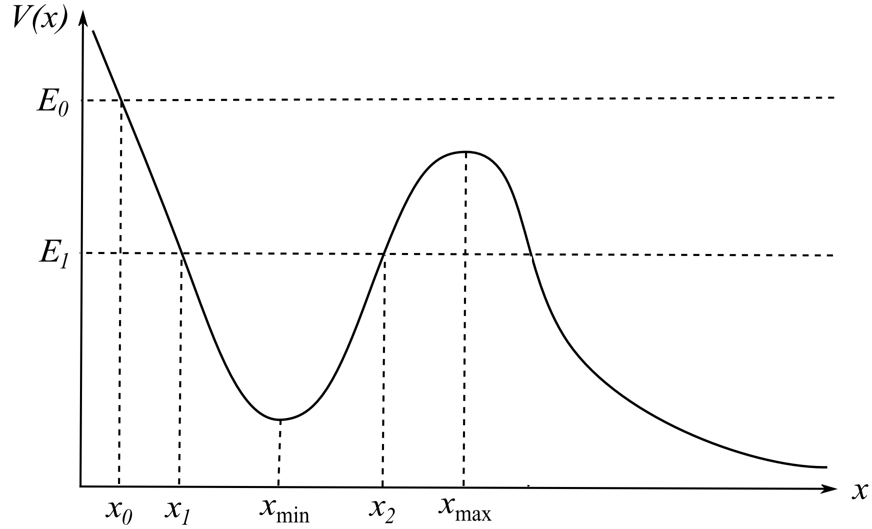


Figure 5: A potential $V(x)$. The force acting on the particle is $F(x) = -V'(x)$.

- Suppose at $t = 0$ the particle starts from rest at x_0. Thus its energy is E_0.

  It experiences a force, and thus accelerates in the positive x-direction as $V'(x_0) < 0$. Thus it gains velocity in the positive x-direction.

  For $t > 0$ its velocity is bound away from zero as $V(x)$ never attains E_0 again. Hence the particle passes x_{max} and $x \to \infty$ as time increases.

- Suppose at $t = 0$ the particle starts from rest at x_1. Thus its energy is E_1.

  It accelerates in the positive x-direction as $V'(x_1) < 0$, gaining a velocity in the x-direction. This velocity is bound away from zero until the approach to $x = x_2$.

  Does the particle reach x_2 in finite time. In particular if the particle is at x_{min} at time t_{min} and it reaches x_2 at time t_2, possibly unbounded, we have

$$t_2 - t_{\text{min}} = \int_{t_{\text{min}}}^{t_2} \mathrm{d}t = \pm \int_{x_{\text{min}}}^{x_2} \frac{\mathrm{d}x}{\left(\frac{2}{m}\right)^{1/2} (E_1 - V(x))^{1/2}}.$$

  The positive root is required as $t_2 > t_{\text{min}}$. Of more interest, the integral is potentially unbounded as the denominator tends to zero as $x \to x_2$ since $V(x_2) = E_1$. If the integral is unbounded then so is t_2 and the particle does not reach x_2 in finite time.

  However, for $x \approx x_2$ we have from a Taylor expansion

$$E_1 - V(x) \approx (x_2 - x)V'(x_2) + \text{higher orders}$$

  and the integrand is of the form

$$\frac{1}{\left(\frac{2}{m}\right)^{1/2} (E_1 - V(x))^{1/2}} = \frac{1}{\left(\frac{2}{m}\right)^{1/2} (V'(x_2))^{1/2} (x_2 - x)^{1/2} (1 + \text{higher orders})^{1/2}},$$

which generates a finite integral, as

$$\int_{x_{\min}}^{x_2} \frac{\mathrm{d}x}{(x_2 - x)^{1/2}} < \infty.$$

Hence t_2 is finite, i.e. the particle reaches x_2 in finite time.

Further, at $x = x_2$ the velocity is zero, and the particle accelerates to the left since $V'(x_2) > 0$; the process thus repeats. Hence the particle simply moves back and forth between x_1 and x_2.

3.3 Motion near equilibrium

We continue our consideration of one-dimensional motion on the x axis.

Definition An *equilibrium configuration* is a solution to Newton's second law (3.1) with $x = x_e =$ constant. Since this implies $\ddot{x} = 0$ for all time t, Newton's second law implies that $F(x_e) = 0$, and there is no net force acting on the particle.

When there is a potential with $F = -\mathrm{d}V/\mathrm{d}x = -V'$ then the equilibrium point x_e is a *critical point* of the potential $V(x)$, with $V'(x_e) = 0$.

Motion near an equilibrium point $x = x_e$. Expanding Newton's second law around $x = x_e$, assuming $F(x)$ is suitably smooth and using $F(x_e) = 0$ yields

$$\begin{aligned} m\ddot{x} &= F(x) = F(x_e) + (x - x_e)F'(x_e) + O((x - x_e)^2) && (3.14) \\ &= (x - x_e)F'(x_e) + O((x - x_e)^2) \ . && (3.15) \end{aligned}$$

where $O((x - x_e)^2)$ means higher order terms, which are not larger than

$$\text{Constant} \times (x - x_e)^2$$

for x sufficiently close to x_e.

We change variables to $\xi \equiv x - x_e$, so that the equilibrium point is now at $\xi = 0$.

Assuming we are sufficiently close to the latter, so that the higher order terms in (3.14) are small, we have the *approximate linear differential equation* for ξ:

$$m\ddot{\xi} = F'(x_e)\xi \ . \qquad (3.16)$$

Definition Equation (3.16) is called the *linearized equation of motion* and its solutions are labelled as *linearized solutions.*

For *any* point of equilibrium in one spatial dimension there are three qualitatively different cases for the behaviour of the linearised solutions, depending on the sign of the constant

$$K \equiv -F'(x_e) \ . \qquad (3.17)$$

- $K > 0$

  With $\omega = \sqrt{K/m} > 0$ we have the simple harmonic oscillator equation

  $$\ddot{\xi} + \omega^2 \xi = 0.$$

  The general solution is $\xi(t) = A\cos(\omega t + \phi)$and $\xi = 0$ is a *point of stable equilibrium*.

  For amplitude A small enough so that it is consistent to ignore the higher order terms in the expansion of the force (3.14), the system executes small oscillations around the equilibrium point.

  The frequency of these oscillations is ω.

- $K < 0$

  With $p = \sqrt{-K/m} > 0$. the linearized equation of motion (3.16) now reads

  $$\ddot{\xi} - p^2 \xi \;\; = \;\; 0\ , \qquad (3.18)$$

  with general solution

  $$\xi(t) \;\; = \;\; A\,\mathrm{e}^{pt} + B\,\mathrm{e}^{-pt}\ , \qquad (3.19)$$

  where A and B are integration constants.

  A generic small displacement and small velocity for the system at time $t = 0$ will have both A and B non-zero, and the solution grows exponentially with t, for both $t > 0$ and $t < 0$.

  Such equilibria are hence termed *unstable*.

  The higher order terms in the Taylor expansion, which we ignored, quickly become relevant.

- $K = 0$

  Finally, if $K = 0$ the first two terms in the Taylor expansion in (3.14) are zero, and we need to expand to higher order to determine what happens (although not in this course!).

We may rephrase all of the above discussion in terms of potentials. We similarly expand

$$V(x) \;\; = \;\; V(x_e) + (x - x_e)V'(x_e) + \frac{1}{2}(x - x_e)^2\, V''(x_e) + O((x - x_e)^3)\ . \qquad (3.20)$$

Without loss of generality we may choose the arbitrary additive constant in V so that $V(x_e) = 0$. Moreover, $V'(x_e) = -F(x_e) = 0$. This means that near equilibrium the potential is approximately quadratic:

$$V_{\mathrm{quad}}(x) \;\; = \;\; \frac{1}{2}K(x - x_e)^2\ , \qquad (3.21)$$

where $K = V''(x_e) = -F'(x_e)$, as in (3.17).

A stable equilibrium point with $K > 0$ is then a *local minimum* of the potential, while an unstable equilibrium point with $K < 0$ is a *local maximum*.

Example From the Examination paper, 2003.

A bead of mass m slides along a smooth, straight horizontal wire which passes through the origin O. The bead is attached to a light, straight elastic spring of natural length l and spring constant k, and the other end of the spring is attached to a fixed point P, which is a distance d vertically above O.

(i) If x denotes the coordinate of the bead, relative to O, explain why the tension in the spring is $\mathbb{T} = k\left(\sqrt{d^2 + x^2} - l\right)$, and show that

$$\ddot{x} = \frac{k}{m} x \left(\frac{l}{\sqrt{d^2 + x^2}} - 1 \right) . \tag{3.22}$$

(ii) Find the equilibrium solutions of this equation, and determine whether they are stable or unstable, distinguishing carefully between the two cases $l < d$ and $l > d$.

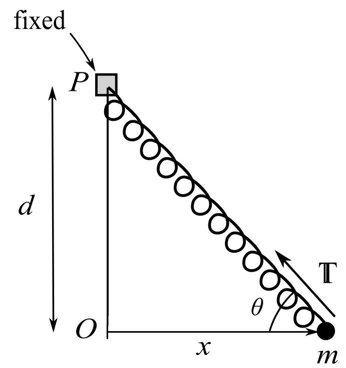


Figure 6: The spring-bead system. The bead of mass m is constrained to move along the x axis.

Solution (i)

The spring is depicted in Fig. 6; its extension from natural length is $\sqrt{d^2 + x^2} - l$.

Thus $\mathbb{T} = k(\sqrt{d^2 + x^2} - l)$, by Hooke's law.

Resolving Newton's second law in the x direction gives

$$m\ddot{x} = -\mathbb{T}\cos\theta = -\frac{\mathbb{T}x}{\sqrt{d^2 + x^2}} = -kx\left(1 - \frac{l}{\sqrt{d^2 + x^2}}\right) \equiv F(x) . \tag{3.23}$$

Dividing both sides by m gives the required equation of motion.

Solution (ii) Equilibrium solutions have the right hand side of (3.23) equal to zero, namely $F(x_e) = 0$ where

$$F(x) = kx \left(\frac{l}{\sqrt{d^2 + x^2}} - 1 \right) . \tag{3.24}$$

The zeros are at $x_e^0 = 0$ and where $l = \sqrt{d^2 + (x_e^{\pm})^2}$ so that $x_e^{\pm} = \pm\sqrt{l^2 - d^2}$. The latter makes sense only if $l \geq d$.

Note also that the configuration is symmetric under taking $x \mapsto -x$, so the behaviour of the two equilibria $x_e^{\pm}$ is the same.

With

$$F'(x) = k \left(\frac{l}{\sqrt{d^2 + x^2}} - 1 \right) - \frac{x^2\, kl}{(d^2 + x^2)^{3/2}} , \tag{3.25}$$

we have

$$F'(x_e^0) = F'(0) \;\;=\;\; k\left(\frac{l}{d} - 1\right) \; . \tag{3.26}$$

Hence the equilibrium at $x_e^0 = 0$ is stable if $l < d$ and unstable if $l > d$.

On the other hand

$$F'(x_e^{\pm}) \;\;=\;\; -\frac{(x_e^{\pm})^2\, kl}{\left(d^2 + (x_e^{\pm})^2\right)^{3/2}} < 0 \; , \tag{3.27}$$

so that $K > 0$.

Hence $x_e^{\pm}$ only exist as equilibria when $l > d$, and if they exist, they are stable.

3.4 Coupled oscillations

So far we have only considered one dimensional systems. In this section we briefly consider the stability of systems in two dimensions.

Suppose we have a dynamical system, i.e. a system of ODEs, described by

$$\ddot{x} \;\;=\;\; F(x,y) \; , \qquad \ddot{y} \;\;=\;\; G(x,y) \; , \tag{3.28}$$

where we shall assume that F and G are suitably smooth and here x and y may general variables, rather than only Cartesian coordinates.

Definition An *equilibrium point* is a solution to (3.28) with $x = x_e$, $y = y_e$ both constant. Thus $F(x_e, y_e) = 0 = G(x_e, y_e)$.

To determine the stability of such an equilibrium point, we again linearize the equations of motion. This means that we write

$$x \;\;=\;\; x_e + \xi \; , \qquad y \;\;=\;\; y_e + \eta \; , \tag{3.29}$$

where ξ and η are small, and then Taylor expand the right hand sides of (3.28), leading to

$$\begin{aligned} \ddot{\xi} \;&=\; F(x_e + \xi, y_e + \eta) \;=\; F(x_e, y_e) + \xi\,\frac{\partial F}{\partial x}(x_e, y_e) + \eta\,\frac{\partial F}{\partial y}(x_e, y_e) + \cdots \; , \\ \ddot{\eta} \;&=\; G(x_e + \xi, y_e + \eta) \;=\; G(x_e, y_e) + \xi\,\frac{\partial G}{\partial x}(x_e, y_e) + \eta\,\frac{\partial G}{\partial y}(x_e, y_e) + \cdots \; , \end{aligned} \tag{3.30}$$

where $\cdots$ denote terms of quadratic and higher order in ξ, η. The *linearized equations of motion* are hence

$$\begin{aligned} \ddot{\xi} \;&=\; a\,\xi + b\,\eta \; , \\ \ddot{\eta} \;&=\; c\,\xi + d\,\eta \; , \end{aligned} \tag{3.31}$$

where we have introduced the constants

$$\begin{aligned} a &= \frac{\partial F}{\partial x}(x_e, y_e) \ , \qquad b = \frac{\partial F}{\partial y}(x_e, y_e) \ , \\ c &= \frac{\partial G}{\partial x}(x_e, y_e) \ , \qquad d = \frac{\partial G}{\partial y}(x_e, y_e) \ . \end{aligned} \tag{3.32}$$

One could solve (3.31) by *e.g.* differentiating the equation for $\ddot{\xi}$ twice, eliminating $\ddot{\eta}$ using its equation and then eliminating for η using the equation for $\ddot{\xi}$. This gives a fourth order ODE in ξ.

However, it is usually more convenient to write (3.31) as a *matrix equation*

$$\begin{pmatrix} \ddot{\xi} \\ \ddot{\eta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \ . \tag{3.33}$$

We then seek solutions to (3.33) of the form

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mathrm{e}^{\lambda t} \ , \tag{3.34}$$

where α, β and λ are constant. Substituting (3.34) into (3.33) and cancelling the overall factor of $\mathrm{e}^{\lambda t}$ gives

$$\lambda^2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ . \tag{3.35}$$

Thus λ^2 is an *eigenvalue* of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with corresponding *eigenvector* $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

The *characteristic equation* is

$$\det\left[\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \lambda^4 - (a+d)\lambda^2 + (ad - bc) = 0 \ , \tag{3.36}$$

which gives the eigenvalues

$$\lambda^2 = \frac{1}{2}\left(a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}\right) . \tag{3.37}$$

For a general system (3.28) the solutions for λ^2 in (3.37) can be complex, in general also leading to complex λ. Note there are two roots for λ^2 and thus four roots for λ.

Remark If λ does not have repeated roots, the general solution is a linear superposition (i.e. a weighted linear sum) of the solutions of the form of Eqn. (3.33), where the summation is over all possible roots for λ.

Remark If λ possesses repeated roots converting the equations to a fourth order ODE in ξ will often be a convenient way of generating the general solution.

Stable and unstable solutions. If **any** of the four roots for λ has positive real part then the solutions have exponential growth and are unstable. If **all** roots for λ have real part less than or equal to zero then the solutions decay or oscillate and are stable.

Definition If all solutions for $\lambda = \pm\lambda_\pm$ given by (3.37) are pure imaginary, we write $\lambda = \pm \mathrm{i}\omega_\pm$, where $\omega_\pm > 0$ are called the *normal frequencies* of the system.

Note as λ pure imaginary for a normal frequency, λ^2 is real and hence the associated eigenvector

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.38}$$

is real. Thus, writing $\mathrm{e}^{\lambda t} = \mathrm{e}^{\pm \mathrm{i}\omega_\pm t}$ in terms of trigonometric functions, the linearized solution is

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix} \cos\left(\omega_+ t + \phi_+\right) + \begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix} \cos\left(\omega_- t + \phi_-\right) \ , \tag{3.39}$$

where

$$\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix}$$

are the eigenvectors corresponding to the eigenvalues $\lambda_\pm^2$, respectively, and $\phi_\pm$ are constants.

A *normal mode* is defined to be the solution for a given eigenvector.

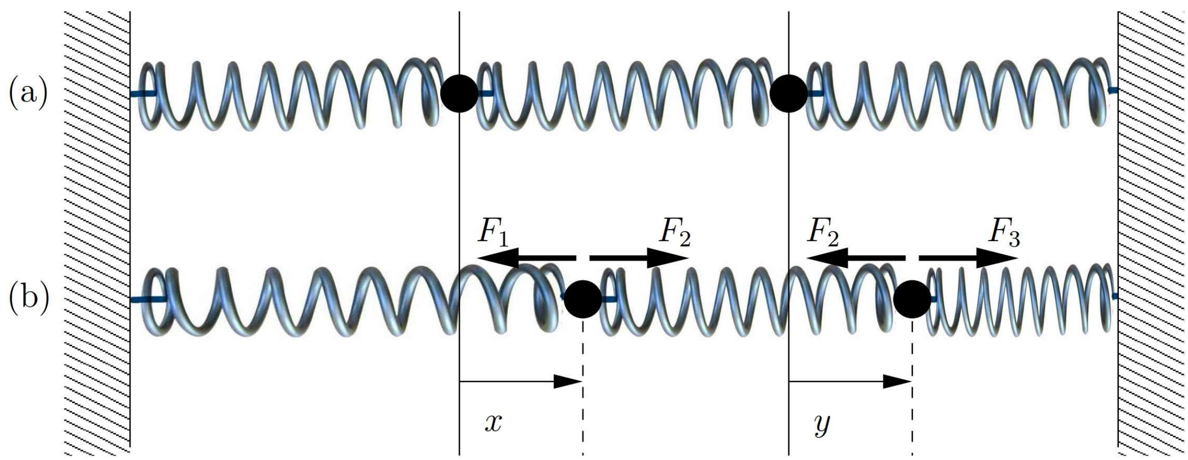


Figure 7: The system of masses and springs. The upper diagram shows the equilibrium configuration, with all springs at natural length l. In the lower diagram the horizontal displacements x and y of the two masses from their equilibrium positions are shown, together with the various Hooke's law forces.

Example:

Consider two particles of mass m attached to three identical springs with spring constant k, as shown in Fig. 7. In the upper plot, the springs are in equilibrium and all at their natural length l. The system is characterised by x, y, as shown with the associated tension forces in the lower plot.

Show that $x = y = 0$ is the only equilibrium point, find the equations of motion, determine the normal frequencies and find the general solution.

Solution By Hooke's law the forces shown in Figure 7 are

$$\mathbf{F}_1 \;=\; -kx\mathbf{i}, \qquad \mathbf{F}_2 \;=\; k(y-x)\mathbf{i} \;\text{ (Left Particle)}, \qquad \mathbf{F}_3 \;=\; -ky\mathbf{i}, \tag{3.40}$$

with $-\mathbf{F}_2$ on the Right Particle given by $-k(y-x)\,\mathbf{i}$.

Resolving Newton's second law in the x-direction for each particle thus gives

$$\begin{aligned} m\ddot{x} &= -kx + k(y-x) = k(y-2x)\;, \\ m\ddot{y} &= -k(y-x) - ky = k(x-2y)\;. \end{aligned} \tag{3.41}$$

We see that the equations are already linear, and that there is a unique equilibrium point at $x = y = 0$.

Thus in this case we may identify $x = \xi$, $y = \eta$. In matrix form (3.41) reads

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} \;=\; \begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\;. \tag{3.42}$$

The characteristic equation is

$$0 = \det\left[\begin{pmatrix} \lambda^2 + \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \lambda^2 + \frac{2k}{m} \end{pmatrix}\right] \;=\; \left(\lambda^2 + \frac{2k}{m}\right)^2 - \left(\frac{k}{m}\right)^2, \tag{3.43}$$

and hence

$$\lambda^2 \;=\; \frac{k}{m}(-2 \pm 1). \tag{3.44}$$

Thus

$$\lambda \;=\; \pm\mathrm{i}\sqrt{\frac{k}{m}}\;, \qquad \pm\mathrm{i}\sqrt{\frac{3k}{m}}\;. \tag{3.45}$$

and the linearized modes (3.34) are oscillatory with normal frequencies

$$\omega_+ = \sqrt{\frac{k}{m}}\;, \qquad \omega_- = \sqrt{\frac{3k}{m}}\;. \tag{3.46}$$

The two values of λ^2 in (3.44) correspond to the two eigenvectors $(1, \pm 1)^{\mathrm{T}}$ of the matrix in (3.42), respectively. Hence the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 1 \end{pmatrix} P\,\cos\left(\sqrt{\frac{k}{m}}t + \varphi\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} Q\,\cos\left(\sqrt{\frac{3k}{m}}t + \psi\right)\;,$$

where P, Q, φ and ψ are constants. The lower frequency ω_+ normal mode has the two masses oscillating together, while the higher frequency ω_- normal mode has the two masses oscillating in opposite directions.

Aside Note that near a stable equilibrium point the system behaves like two *independent* one-dimensional harmonic oscillators, of frequencies $\omega_\pm$.

4 Motion in higher dimensions

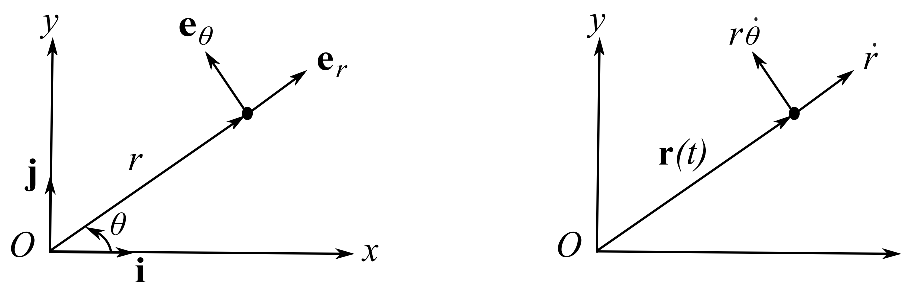
In this section we develop some general formalism that is useful for analysing dynamics in two and three dimensions.

In particular we introduce *velocity and acceleration* in plane polar coordinates and *angular momentum*, together with *conservative forces*, *central forces*.

The dynamics for each of these two types of force leads to a *conserved quantity*, *i.e.* a quantity that is constant during the motion.

4.1 Planar motion in polar coordinates

Motion in a plane is sometimes conveniently described using polar coordinates.



(a) Cartesian and polar coordinates. (b) Velocity in polar coordinates.

Figure 8

Recall that Cartesian coordinates (x, y) are related to polar coordinates (r, θ) by (Fig. 8a)

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,, \tag{4.1}$$

so that

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} y/x, \quad r \geq 0, \quad \theta \in [0, 2\pi).$$

Definition

- $\mathbf{e}_r$ is *the unit vector in direction of increasing* r.
- $\mathbf{e}_\theta$ *is unit vector in direction of increasing* θ.

Hence (Fig. 8a)

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \qquad \mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \quad \mathbf{e}_r \cdot \mathbf{e}_\theta = 0, \tag{4.2}$$

Except at the origin $\{\mathbf{e}_r,\ \mathbf{e}_\theta\}$ form an orthonormal basis – it is important to note that $\mathbf{e}_r,\ \mathbf{e}_\theta$ vary in space and, in particular, are functions of θ.

The position of a particle is simply given by

$$\mathbf{r} = (x, y) = r\,\mathbf{e}_r.$$

For a time-dependent trajectory $\mathbf{r}(t)$ of the particle we thus have

$$\dot{\mathbf{r}} \;=\; \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r \ . \tag{4.3}$$

But from (4.2) we have

$$\begin{aligned} \dot{\mathbf{e}}_r &= -\dot{\theta}\sin\theta\,\mathbf{i} + \dot{\theta}\cos\theta\,\mathbf{j} \;=\; \dot{\theta}\,\mathbf{e}_\theta \ , \\ \dot{\mathbf{e}}_\theta &= -\dot{\theta}\cos\theta\,\mathbf{i} - \dot{\theta}\sin\theta\,\mathbf{j} \;=\; -\dot{\theta}\,\mathbf{e}_r \ . \end{aligned} \tag{4.4}$$

Hence we have the velocity of the particle can be written as (see Figure 8b).

$$\dot{\mathbf{r}} \;=\; \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta \ , \tag{4.5}$$

where $\dot{\theta}$ is referred to as the *angular velocity* of the particle.

The second term has arisen because $\{\mathbf{e}_r,\ \mathbf{e}_\theta\}$ depend on θ and thus inherit the time-dependence of $\theta(t)$ on the trajectory $\mathbf{r}(t)$.

We may find the analogous expression for acceleration by taking another time derivative, using (4.4):

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\,\mathbf{e}_\theta + r\ddot{\theta}\,\mathbf{e}_\theta + r\dot{\theta}\,\dot{\mathbf{e}}_\theta \ , \\ &= (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\mathbf{e}_\theta \ , \\ &= (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\,\mathbf{e}_\theta \ , \end{aligned} \tag{4.6}$$

using $2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})$ in the final equality.

Example (Uniform circular motion): Consider a particle moving in a circle of radius R, centre the origin O, at constant speed v. Since $r = R =$ constant we have $\dot{r} = 0$. Thus from (4.5) its velocity is

$$\dot{\mathbf{r}} \;=\; R\,\dot{\theta}\,\mathbf{e}_\theta \ . \tag{4.7}$$

This is tangent to the circle, with speed is $v = |\dot{\mathbf{r}}|$,

Hence $v = R|\dot{\theta}|$, and the angular speed $|\dot{\theta}| = \frac{v}{R}$ is constant.

Since $\dot{\theta}$ is constant, $\ddot{\theta} = 0$, and similarly $\dot{r} = 0$ so that $\ddot{r} = 0$.

Thus from (4.6) the acceleration is

$$\ddot{\mathbf{r}} \;=\; -R\,\dot{\theta}^2\,\mathbf{e}_r \;=\; -\frac{v^2}{R}\,\mathbf{e}_r \ , \tag{4.8}$$

which may be summarised as

acceleration in circular motion is $\dfrac{v^2}{r}$ towards the centre of the circle O.

Newton's second law implies that in order to generate this acceleration we need a force of magnitude $F = mv^2/R = mR\,\dot{\theta}^2$ directed towards the origin – this is called the *centripetal force*.

4.2 Conservative forces

In section 3.1 we saw that for motion in one dimension and forces $F = F(x)$ there is a conserved energy.

In three dimensions this is no longer necessarily the case: we need an additional constraint on $\mathbf{F} = \mathbf{F}(\mathbf{r})$ in order for energy to be conserved.

One might anticipate this: energy is a scalar quantity, and without any further input there is no natural way to construct a scalar from the vector $\mathbf{F}$, analogously to the one dimensional case.

Definition The *kinetic energy* of a point particle with trajectory $\mathbf{r}(t)$ is

$$T = \frac{1}{2}m|\dot{\mathbf{r}}|^2,$$

where $\mathbf{r}(t)$ is the particle's position in an inertial frame.

We then have the following important result:

Conservation of Energy The quantity

$$E \;=\; T + V \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(\mathbf{r}) \;, \tag{4.9}$$

is *conserved* for a point particle trajectory if the force $\mathbf{F} = \mathbf{F}(\mathbf{r})$ takes the form

$$\mathbf{F} \;=\; -\nabla V \;. \tag{4.10}$$

That is, in Cartesian coordinates $\mathbf{F} = (-\partial_x V, -\partial_y V, -\partial_z V)$.

Proof: From Newton's second law

$$m\ddot{\mathbf{r}} \;=\; \mathbf{F}(x) = -\nabla V, \tag{4.11}$$

where $\mathbf{r} = \mathbf{r}(t)$ is a function of time. Taking the dot product of both sides by $\dot{\mathbf{r}}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{\mathbf{r}}^2\right) = m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = -\dot{\mathbf{r}}\cdot\nabla V = -\frac{\mathrm{d}V(\mathbf{r}(t))}{\mathrm{d}t},$$

with the first and last equalities from the product and chain rules of differentiation respectively. Hence

$$\frac{1}{2}m\dot{\mathbf{r}}^2 + V(\mathbf{r}) = E,\ \text{constant}\ . \tag{4.12}$$

■

To further understand the condition (4.10) we generalize the notion of work to three dimensions:

Definition The *work done* by a force $\mathbf{F}$ in moving a particle from $\mathbf{r}_1$ to $\mathbf{r}_2$ along a curve C is

$$W \;=\; \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}\ . \tag{4.13}$$

In contrast to the definition in 1D (3.3), the line integral (4.13) in higher dimensions depends on the precise curve C, and not just on its endpoints $\mathbf{r}_1$, $\mathbf{r}_2$.

If we now suppose that $\mathbf{r}(t)$ is the trajectory of a particle satisfying Newton's second law, starting at position $\mathbf{r}_1 = \mathbf{r}(t_1)$ and ending at $\mathbf{r}_2 = \mathbf{r}(t_2)$, then we may write

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}} \, \mathrm{d}t \;=\; m \int_{t_1}^{t_2} \ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} \, \mathrm{d}t \;=\; \frac{m}{2} \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} |\dot{\mathbf{r}}|^2 \, \mathrm{d}t \;=\; T(t_2) - T(t_1). \tag{4.14}$$

Thus, as in one dimension, the work done by the force is the change in kinetic energy.

Suppose now that the total energy E given by (4.9) is conserved, so that $E = T(t_1) + V(\mathbf{r}_1) = T(t_2) + V(\mathbf{r}_2)$. Hence (4.14) implies that

$$W \;=\; \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \;=\; V(\mathbf{r}_1) - V(\mathbf{r}_2) \; . \tag{4.15}$$

The right hand side depends only on the endpoints $\mathbf{r}_1$, $\mathbf{r}_2$ of the curve C.

Hence if energy is conserved, then the work done is *independent* of the choice of curve C connecting $\mathbf{r}_1$ to $\mathbf{r}_2$.

In the Prelims Multivariable Calculus course you prove that if this is true for *all* curves C then $\mathbf{F}$ takes the form (4.10).[1]

Thus if energy is always conserved then there is a function V such that $\mathbf{F} = -\nabla V$.

Definition A force $\mathbf{F} = \mathbf{F}(\mathbf{r})$ is said to be *conservative* if there exists a *potential energy* function $V = V(\mathbf{r})$ such that

$$\mathbf{F} = -\nabla V \; . \tag{4.16}$$

Note that as in one dimension the potential V is only defined up to an additive constant.

Examples:

(i) Any *constant* force $\mathbf{F}_{\text{const}}$ is conservative, with potential $V(\mathbf{r}) = -\mathbf{F}_{\text{const}} \cdot \mathbf{r}$.

For gravity, $\mathbf{F} = -mg\,\mathbf{k}$, the corresponding potential function is simply $V(\mathbf{r}) = mg\,\mathbf{k} \cdot \mathbf{r} = mgz$.

(ii) In section 6.1 we will show that any force of the form $\mathbf{F} = F(|\mathbf{r}|)\,\mathbf{e}_r$ is conservative, where $\mathbf{e}_r = \mathbf{r}/|\mathbf{r}|$.

Conservative forces enjoy the following equivalent definitions:

Theorem (Lectured in Prelims Multivariable Calculus) Let $\mathbf{F} : S \to \mathbb{R}^3$ be a vector field, where the domain $S \subset \mathbb{R}^3$ is open and path connected. Then the following three statements are equivalent:

1. $\mathbf{F}$ is conservative, *i.e.* there exists a potential $V : S \to \mathbb{R}$ such that $\mathbf{F} = -\nabla V$.

[1] See the Theorem below.

2. Given any two points $\mathbf{r}_1$, $\mathbf{r}_2$ in S, and any curve C in S starting at $\mathbf{r}_1$ and ending at $\mathbf{r}_2$, then the integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of the choice of C.

3. For any simple closed curve C in S we have $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.

It is also shown in Multivariable Calculus that conservative forces satisfy $\nabla \wedge \mathbf{F} = \mathbf{0}$, although we will not need this fact.

4.3 Central forces and angular momentum

Another important concept is that of a *central force*:

Definition A force that is always directed along the line joining a particle to a fixed position in an inertial frame is called a *central force*.

Without loss, choose this point as the origin of the frame and hence

$$\mathbf{F} \propto \mathbf{r} \, , \tag{4.17}$$

where $\mathbf{r}$ is the position vector of the particle, measured from the origin O.

The importance of central forces is that they always lead to an associated conserved vector quantity.

Definition The *angular momentum* $\mathbf{L} = \mathbf{L}_P$ of a particle about a point P in an inertial frame is the *moment* of linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ *about* P. That is,

$$\mathbf{L}_P \equiv (\mathbf{r} - \mathbf{x}) \wedge \mathbf{p} = (\mathbf{r} - \mathbf{x}) \wedge m\dot{\mathbf{r}} \, . \tag{4.18}$$

Here $\mathbf{x}$ is the position vector of the point P, while $\mathbf{r}$ is the position of the particle (both measured from the origin O).

It is important to note that $\dot{\mathbf{r}}$ is the velocity of the particle in the inertial frame, *not* the velocity relative to P, which in general may be moving, $\mathbf{x} = \mathbf{x}(t)$.

This definition makes it clear that the angular momentum depends on the point P. However, for central forces there is a natural choice for P, namely $P = O$, the centre of the force.

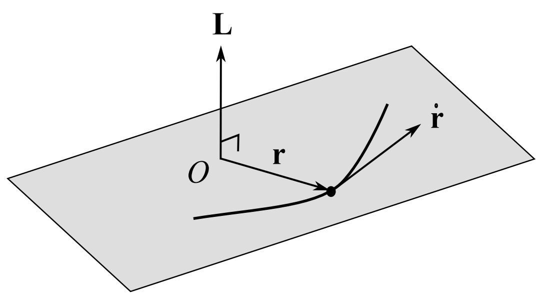


Figure 9: The planar motion of a particle acted on by a central force, with centre O. When the conserved angular momentum $\mathbf{L}$ is non-zero $\mathbf{L}$ is normal to the plane of motion through O.

Proposition If a particle is acted on by a central force with centre O then the angular momentum $\mathbf{L} = \mathbf{L}_O$ is conserved, and the path of the particle lies entirely in a fixed plane through O. That is, the motion is *planar*.

Proof: We have

$$\dot{\mathbf{L}} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \wedge m\dot{\mathbf{r}}) = \dot{\mathbf{r}} \wedge m\dot{\mathbf{r}} + \mathbf{r} \wedge m\ddot{\mathbf{r}} = \mathbf{r} \wedge m\ddot{\mathbf{r}} = \mathbf{r} \wedge \mathbf{F} = \mathbf{0} \tag{4.19}$$

where we have used $\mathbf{a} \wedge \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ and Newton's second law, while the last equality holds since $\mathbf{F} \propto \mathbf{r}$ for a central force.

Thus $\mathbf{L}$ is constant.

- Suppose $\mathbf{L} \neq \mathbf{0}$. As seen in Prelims Geometry, $\mathbf{r} \cdot \mathbf{L} = 0$ entails $\mathbf{r}$ is in the plane containing the origin and perpendicular to $\mathbf{L}$ (see Figure 9).

  In addition $\dot{\mathbf{r}} \cdot \mathbf{L} = 0$ entails the velocity is always in this plane.

  This means that the motion is confined to the plane through O with normal vector $\mathbf{L}$.

- $\mathbf{L} = \mathbf{r} \wedge m\dot{\mathbf{r}} = \mathbf{0}$, a degenerate case. Then either the position, $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ are parallel, *where we include this to mean that one or both of these vectors are zero.*

  If $\dot{\mathbf{r}}$ is zero the particle rests at a fixed point; otherwise the particle trajectory is on a straight line through the origin, both of which are (degenerate) examples of motion restricted to a plane containing the origin.

■

Suppose that $\mathbf{L} = \mathbf{L}_O$ is conserved, as in the last Proposition. In particular the direction of $\mathbf{L}$ is constant, and we may choose this as the z direction, so that $\mathbf{L}_O \propto \mathbf{k}$.

Introducing polar coordinates for the planar motion in the (x, y) plane, we have

$$\begin{aligned} \mathbf{L} \;=\; \mathbf{r} \wedge m\dot{\mathbf{r}} &= r\,\mathbf{e}_r \wedge m\left(\dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta\right) , \\ &= mr^2\dot{\theta}\,\mathbf{k} , \end{aligned} \tag{4.20}$$

where we have used (4.5) in the first line, and $\mathbf{k} = \mathbf{i} \wedge \mathbf{j} = \mathbf{e}_r \wedge \mathbf{e}_\theta$ in the last step.

This proves the following result, which will be important later:

Proposition If angular momentum $\mathbf{L}$ is conserved, then the quantity

$$h \;\equiv\; r^2\dot{\theta} \;\equiv\; \text{angular momentum per unit mass} \tag{4.21}$$

is conserved, where (r, θ) are polar coordinates in the plane of motion.

For completeness we conclude this section with a definition and brief discussion of *torque*, though it will not be required until Section 7.

Definition The *torque* $\boldsymbol{\tau} = \boldsymbol{\tau}_P$ of a force $\mathbf{F}$, about a point P with position vector $\mathbf{x}$, acting on a particle with position vector $\mathbf{r}$ is

$$\boldsymbol{\tau}_P \;\equiv\; (\mathbf{r} - \mathbf{x}) \wedge \mathbf{F} . \tag{4.22}$$

In other words, the torque is the *moment* of the force about P. The direction of $\boldsymbol{\tau}_P$ is normal to the plane containing $\mathbf{r} - \mathbf{x}$ and $\mathbf{F}$, and may be regarded as the *axis* about which the force tends to rotate the particle about P.

If P is a fixed point in the inertial frame, so that $\mathbf{x} =$ constant, then using (4.18) and Newton's second law we have

$$\dot{\mathbf{L}}_P \;=\; (\mathbf{r} - \mathbf{x}) \wedge m\ddot{\mathbf{r}} \;=\; (\mathbf{r} - \mathbf{x}) \wedge \mathbf{F} \;=\; \boldsymbol{\tau}_P , \tag{4.23}$$

and the torque is the rate of change of angular momentum.

This can be compared with Newton's second law itself, written in the form $\dot{\mathbf{p}} = \mathbf{F}$, which says that the force is the rate of change of linear momentum.

5 Constrained systems

In this section we consider *constrained* dynamical systems, for example: beads threaded on smooth wires and marbles rolling in smooth dishes.

If a particle is going to be constrained to move on a particular curve or surface in $\mathbb{R}^3$, there must be a constraint force ensuring this, and this often the Normal Reaction Force though it can also be string Tension as we will detail below

The dynamics happens in $\mathbb{R}^3$, but the constraints effectively reduce the motion to a one-dimensional or two-dimensional dynamical system.

5.1 Constraint forces

Below we label the constraint force by $\mathbf{N}$, for normal, i.e. perpendicular. In particular:

Assumption: The constraint force $\mathbf{N}$ is always *perpendicular* to the constraint space.

Since by definition the velocity of the particle $\dot{\mathbf{r}}$ is always *tangent* to the constraint space, we have

$$\mathbf{N} \cdot \dot{\mathbf{r}} \ = \ 0 \ . \tag{5.1}$$

The *work done* by the force $\mathbf{N}$ when the particle moves along a curve C in the constraint space is defined as definition (Eqn. (4.13))

$$W \ = \ \int_C \mathbf{N} \cdot \mathrm{d}\mathbf{r} \ = \ \int \mathbf{N} \cdot \dot{\mathbf{r}} \, \mathrm{d}t \ = \ 0 \ . \tag{5.2}$$

Thus such constraint forces *do no work* during the constrained motion of the particle.

Note it is assumed there is no component of the constraint force *tangent* to the constraint space. Thus the assumption also requires the absence of friction with the surface associated with the constraint force.

If we consider a particle of mass m, acted on by a force $\mathbf{F}_0$, that is then further constrained to move on a smooth constraint space, Newton's second law simply reads

$$m\ddot{\mathbf{r}} \ = \ \mathbf{F} \ = \ \mathbf{F}_0 + \mathbf{N} \ , \tag{5.3}$$

where $\mathbf{N}$ is the normal reaction/constraint force. We have the following important result:

Conservation of Energy Theorem (Constrained motion) Suppose that the force $\mathbf{F}_0 = -\nabla V$ is conservative, with potential $V = V(\mathbf{r})$. Then the total energy $E = T + V$ for the *constrained* motion of a point particle is conserved.

Proof: From Newton's second law

$$m\ddot{\mathbf{r}} \;=\; \mathbf{F}_0(x) + \mathbf{N} = -\nabla V(\mathbf{r}) + \mathbf{N}, \tag{5.4}$$

where $\mathbf{r} = \mathbf{r}(t)$ is a function of time. Taking the dot product of both sides by $\dot{\mathbf{r}}$ on noting $\mathbf{N}\cdot\dot{\mathbf{r}} \;=\; 0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{\mathbf{r}}^2\right) = m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = -\dot{\mathbf{r}}\cdot\nabla V = -\frac{\mathrm{d}V(\mathbf{r}(t))}{\mathrm{d}t},$$

and, once more, we have energy conservation:

$$\frac{1}{2}m\dot{\mathbf{r}}^2 + V(\mathbf{r}) = T + V = E, \text{ constant }. \tag{5.5}$$

■

5.2 The simple pendulum

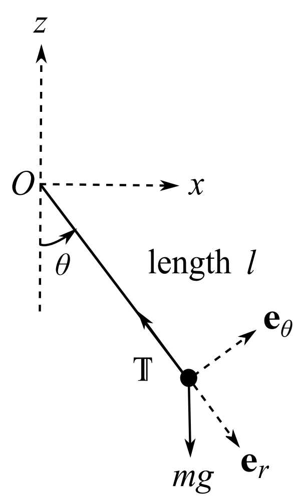


Figure 10: A simple pendulum.

Consider the *simple pendulum.* This consists of a mass m fixed to the end of a light (*i.e.* negligible mass) rod of length l. The other end of the rod is hinged smoothly at a point O and is free to swing in a vertical plane under gravity, Fig. 10.

The rod constrains the mass m to move on a circle of radius l in the (z, x) plane, centred on the pivot point O. The constraint space in this case is hence a circle. See Figure 10

The constraint force for the motion is the *tension* $\mathbb{T}$ in the rod.

Given that the motion will lie on a circle, it is useful to introduce polar coordinates in the (z, x) plane: $z = -l\cos\theta$, $x = l\sin\theta$. The corresponding unit vectors are

$$\mathbf{e}_r \;=\; -\cos\theta\,\mathbf{k} + \sin\theta\,\mathbf{i}\;, \qquad \mathbf{e}_\theta \;=\; \sin\theta\,\mathbf{k} + \cos\theta\,\mathbf{i}\;. \tag{5.6}$$

Although these are marginally different to the polar coordinates in the (x, y) plane in Figure 8a, the essential point is that as in (4.4) we again have $\dot{\mathbf{e}}_r = \dot{\theta}\,\mathbf{e}_\theta$, $\dot{\mathbf{e}}_\theta = -\dot{\theta}\,\mathbf{e}_r$.

It follows that the velocity and acceleration are again given by (Eqns. (4.5), (4.6))

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta \ , \qquad \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\,\mathbf{e}_\theta \ , \tag{5.7}$$

where $\mathbf{r} = (z, x)$.

The forces acting on the mass m are gravity and the constraint force: in the notation of section 5.1 we have

$$\mathbf{F}_0 \ = \ -mg\,\mathbf{k} \ , \qquad \mathbf{N} \ = \ -\mathbb{T}\,\mathbf{e}_r \ , \tag{5.8}$$

where the total force acting is $\mathbf{F} = \mathbf{F}_0 + \mathbf{N}$.

Newton's second law (5.3) is a vector equation. Resolving in the $\mathbf{e}_\theta$ direction gives

$$\frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = m\ddot{\mathbf{r}}\cdot\mathbf{e}_\theta \ = \ \mathbf{F}\cdot\mathbf{e}_\theta \ = \ -mg\,\sin\theta \ , \tag{5.9}$$

where the first equality is from Eqn. (5.7).

However, here $r = l$ is constant, so that (5.9) reads $ml\ddot{\theta} = -mg\sin\theta$, which rearranges to

$$\ddot{\theta} \ = \ -\frac{g}{l}\sin\theta \ , \tag{5.10}$$

the equation of motion for the simple pendulum.

Resolving in the $\mathbf{e}_r$ direction gives

$$-ml\dot{\theta}^2 = m\ddot{r} - m\dot{\theta}^2 = m\ddot{\mathbf{r}}\cdot\mathbf{e}_r \ = \ \mathbf{F}\cdot\mathbf{e}_r \ = \ mg\,\cos\theta - \mathbb{T} \ . \tag{5.11}$$

Thus

$$\mathbb{T} \ = \ ml\dot{\theta}^2 + mg\cos\theta \ . \tag{5.12}$$

This says that the tension $\mathbb{T}$ balances the component of the weight along the rod $mg\cos\theta$, and the centripetal force $ml\dot{\theta}^2$ for circular motion about the origin O.

We cannot solve the equation of motion (5.10) in closed form, as simple as it looks, but we can look at the *equilibrium configurations*, and *conservation of energy.*

Equilibria: Notice there are two equilibrium configurations, where the right hand side of (5.10) is zero: $\theta = 0$ and $\theta = \pi$.

The former has the pendulum hanging down vertically, and for small oscillations (*i.e.* small θ) we may approximate $\sin\theta \simeq \theta$. In this linearized approximation (5.10) becomes the simple harmonic motion

$$\ddot{\theta} \ = \ -\omega^2\theta \ , \qquad \text{where} \qquad \omega^2 \ = \ \frac{g}{l} > 0 \ . \tag{5.13}$$

Thus, as is intuitively obvious, $\theta = 0$ is a *stable equilibrium.*

For small oscillations about this point the pendulum executes simple harmonic motion with angular frequency ω, so that

$$\theta(t) = A\sin(\omega t) + B\sin(\omega t)$$

which has the period

$$T \;=\; 2\frac{\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}\ . \tag{5.14}$$

To consider the second equilibrium position, $\theta = \pi$, we set $\theta = \pi + \xi(t)$, with $\xi(t)$ small, so that $\sin\theta = \sin(\pi+\xi) \simeq -\sin\xi \simeq -\xi$. This yields

$$\ddot{\xi} \;=\; -\frac{g}{l}(-\xi) \;=\; \frac{g}{l}\xi\ . \tag{5.15}$$

The general solution is $\xi(t) = C\,\mathrm{e}^{\sqrt{g/l}\,t} + D\,\mathrm{e}^{-\sqrt{g/l}\,t}$, and the equilibrium is unstable.

Conservation of energy: Total energy is conserved, as the gravitational force $\mathbf{F}_0 = -mg\,\mathbf{k}$ is conservative, with potential $V(\mathbf{r}) = V(x,y,z) = mgz$. The total energy is

$$E \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(\mathbf{r}) \;=\; \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta\ . \tag{5.16}$$

Thus we have $E \geq -mgl$, with equality for the stable equilibrium at $\theta = 0$. However, if $E > mgl$ then $\cos\theta_0 = -E/mgl$ has no solution, and hence $\dot{\theta}$ is never zero. In this case the system has so much energy that the pendulum swings over the top of the pivot point.

Aside As in section 3.2, we may view (5.16) as a *first order* ODE for $\theta(t)$, and integrate it. Rerranging we have

$$\dot{\theta}^2 \;=\; \frac{2E}{ml^2} + \frac{2g}{l}\cos\theta\ , \tag{5.17}$$

which integrates to

$$t \;=\; \pm\int\frac{\mathrm{d}\theta}{\sqrt{2E/ml^2 + 2(g/l)\cos\theta}}\ . \tag{5.18}$$

If we assume that the pendulum starts at $\theta = 0$ at time $t = 0$, and reaches a maximum angle of $\theta_0 > 0$ in its swing, then we may compute the period of the swing:

$$T \;=\; 4\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{2E/ml^2 + 2(g/l)\cos\theta}} \;=\; 4\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{2\cos\theta - 2\cos\theta_0}}\ . \tag{5.19}$$

Here we have noted that at the top of the swing $\dot{\theta} = 0$, and hence from (5.17) $\cos\theta_0 = -E/mgl$. The factor of 4 in (5.19) arises because the integral from 0 to θ_0 is only a quarter of one period.

Aside Compare to the result for small oscillations (5.14). One can derive this from the general formula (5.19) by making the approximation $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$ in the integral. More generally the integral in (5.19) is an *elliptic integral*. We also see that the period T is a dimensionless number times $\sqrt{l/g}$, where the dimensionless number in general depends on the initial conditions (via the conserved energy E).

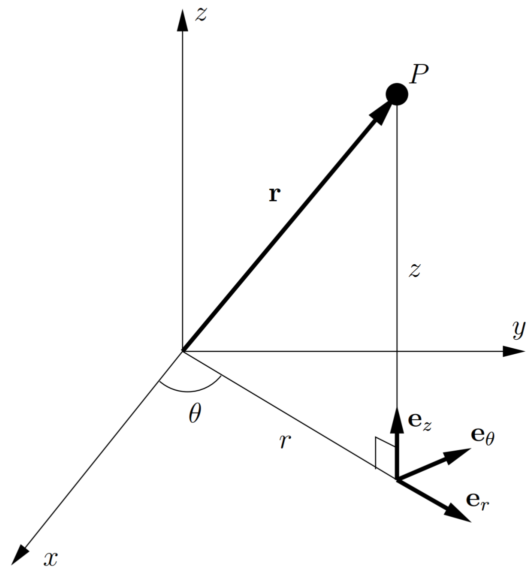


Figure 11: A particle P is shown on a 3-dimensional Cartesian graph. The relationship between the polar coordinates and the Cartesian coordinates is shown.

5.3 Motion on a surface of revolution under gravity

First we introduce a useful coordinate system, Cylindrical polars, before considering the motion of a particle under gravity on a frictionless surface of revolution.

5.3.1 Cylindrical Polars

Cylindrical polar co-ordinates are (r, θ, z). The relationship between cylindrical polars and cartesians (x, y, z) is given by (Figure 11)

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{5.20}$$

The position vector is given by

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z$$

where $\mathbf{e}_z = \mathbf{k}$ is the unit vector in the direction of increasing z.

Note that now $|\mathbf{r}| \neq r$. By Pythagoras' theorem we have instead $|\mathbf{r}| = (r^2 + z^2)^{1/2}$. The particle's velocity is

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta + \dot{z}\,\mathbf{e}_z.$$

The acceleration is

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\,\mathbf{e}_\theta + \ddot{z}\,\mathbf{e}_z.$$

These are the same as in plane polar coordinates but with the simple Cartesian terms relating to z as $\mathbf{e}_z$ does not vary in space.

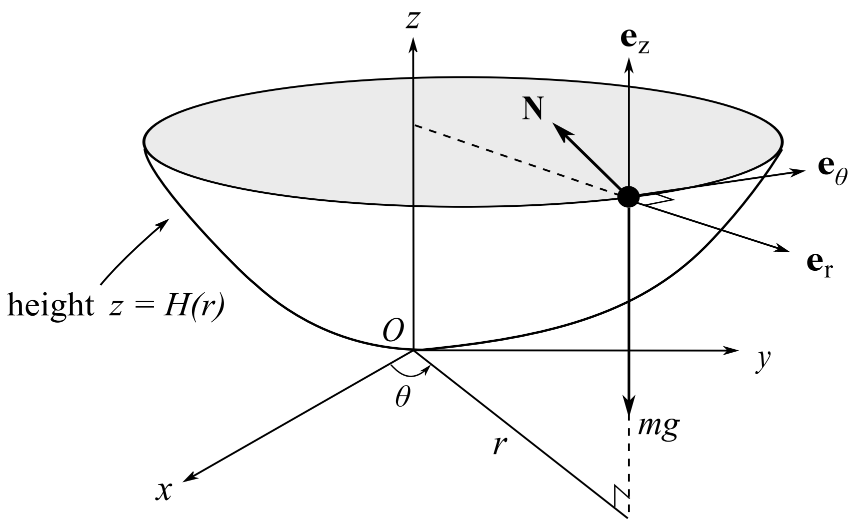


Figure 12: A particle on a surface of revolution about the z-axis. The surface is given by $z = H(r)$. The forces acting on the particle are $m\mathbf{g}$ (weight) and $\mathbf{N}$, the normal reaction.

5.3.2 A mass m moving under gravity on a smooth surface of revolution

Consider a mass m moving under gravity on a frictionless smooth surface of revolution, as in Fig. 12, with height z given by $z = H(r)$.

When there is no friction the only force exerted by the surface on the particle is the normal reaction force $\mathbf{N}$ perpendicular to the surface. The total force on the particle is therefore

$$\mathbf{F} = -mg\mathbf{e}_z + \mathbf{N}.$$

The surface of revolution or "bowl" has z as a symmetry axis, i.e. its equation is $r = r(z)$ rather than $r = r(z, \theta)$. Since this is independent of θ the resulting surface will be invariant under rotation about the z axis, which rotates the θ coordinate. This also implies that $\mathbf{e}_\theta$ is *tangent* to the surface at every point, and hence in particular we have $\mathbf{N} \cdot \mathbf{e}_\theta = 0$.

Newton's second law (5.3) thus reads

$$m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\,\mathbf{e}_z\right] = m\ddot{\mathbf{r}} = \mathbf{F} \;=\; -mg\,\mathbf{e}_z + \mathbf{N}\;. \tag{5.21}$$

Since there is no θ-component of force we have

$$0 = \mathbf{e}_\theta \cdot m\ddot{\mathbf{r}} = \frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}),$$

so that $mr^2\dot{\theta} = mh =$ constant, where $h = r^2\dot{\theta}$ by definition. Hence

$$\mathbf{e}_z \cdot (\mathbf{r} \wedge m\dot{\mathbf{r}}) = m\mathbf{e}_z \cdot ((r\mathbf{e}_r + z\mathbf{e}_z) \wedge (\dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta + \dot{z}\,\mathbf{e}_z)) = mr^2\dot{\theta} = mh = \text{Const},$$

and there is conservation of angular momentum about the z-axis.

Exercise Use the equations of motion to determine the angular momentum in the z-direction, as a function of r, for a particle rotating around the surface with $\dot{r} = 0$.

We have the angular momentum in the z-direction is $mh = mr^2\dot{\theta}$ and $\dot{z} = 0$ as $z = H(r)$ and $\dot{r} = 0$. Thus $\ddot{z} = \ddot{r} = 0$ and we have the equation of motion reduces to

$$0 = -r\dot{\theta}^2\,\mathbf{e}_r + g\,\mathbf{e}_z - \frac{\mathbf{N}}{m} \;= -\frac{h^2}{r^3}\,\mathbf{e}_r + g\,\mathbf{e}_z - \frac{\mathbf{N}}{m}, \tag{5.22}$$

Resolving in any direction other than a tangential one will be complicated by components of $\mathbf{N}$. We have resolved in the $\mathbf{e}_\theta$ direction already, so we need to resolve using another independent tangent vector.

We have $f(r, z, \theta) = z - H(r) = 0$ on the surface so a normal is[2]

$$\mathbf{n} \;=\; \nabla f \;=\; \mathbf{e}_z - H'(r)\mathbf{e}_r \tag{5.23}$$

and hence

$$\mathbf{t} = H'(r)\mathbf{e}_z + \mathbf{e}_r\ .$$

is a tangent vector that is perpendicular to both the normal and $\mathbf{e}_\theta$.

Resolving in this direction will eliminate the normal reaction force $\mathbf{N}$ and gives

$$0 = \mathbf{t}\cdot\left[-\frac{h^2}{r^3}\,\mathbf{e}_r + g\,\mathbf{e}_z - \frac{\mathbf{N}}{m}\right] = -\frac{h^2}{r^3} + gH'(r), \tag{5.24}$$

and hence the angular momentum is $mh = m(gH'(r)r^3)^{1/2}$.

Aside Resolving

$$m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\,\mathbf{e}_z\right] = m\ddot{\mathbf{r}} = \mathbf{F} \;=\; -mg\,\mathbf{e}_z + \mathbf{N}\ . \tag{5.25}$$

in the $\mathbf{t}$ direction and eliminating z via $z = H(r)$ and eliminating $\dot{\theta}$ using $r^2\dot{\theta} = h$ will generate an equation of motion for $r(t)$ for any general motion. If this is required, it can usually be derived more easily via the conservation of energy.

5.3.3 Conservation of energy

From conservation of energy we have

$$E \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 + mgz \;=\; \text{constant}\ . \tag{5.26}$$

In cylindrical polars

$$\dot{\mathbf{r}}^2 \;=\; \dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = |\dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta + \dot{z}\,\mathbf{e}_z|^2 \;=\; \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 \tag{5.27}$$

while $\dot{\theta} = h/r^2(t)$ and $z = H(r(t))$, so that $\dot{z} = H'(r)\dot{r}$ by the chain rule.

[2]The gradient in cylindrical polar coordinates is $\nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{e}_z$. See Multivariable Calculus.

Hence

$$E = \frac{1}{2}m\left[\dot{r}^2 + \frac{h^2}{r^2} + \left(H'(r)\right)^2\dot{r}^2\right] + mg\,H(r)\ . \tag{5.28}$$

This is the first integral of the equation for $r(t)$. Thus differentiate in time for the equation of motion for $r(t)$. Thus on factoring a $m\dot{r}$ we have

$$m\dot{r}\left\{\left[1 + \left(H'(r)\right)^2\right]\ddot{r} + H'(r)H''(r)\,\dot{r}^2 - \frac{h^2}{r^3} + g\,H'(r)\right\} = 0\ . \tag{5.29}$$

Providing $\dot{r} \neq 0$, dividing by $m\dot{r}$ gives

$$\left\{\left[1 + \left(H'(r)\right)^2\right]\ddot{r} + H'(r)H''(r)\,\dot{r}^2 - \frac{h^2}{r^3} + g\,H'(r)\right\} = 0\ , \tag{5.30}$$

the equation of motion for $r(t)$.

Aside If $\dot{r}$ is zero at a point in time, Eqn (5.30) is still valid at this point by the continuity of all of its terms. Eqn (5.30) still holds if $\dot{r} = 0$ all time, as may be confirmed by comparison with the final identity of Eqn (5.24) and as expected since the trajectory with $\dot{r} = 0$ is the limit of a trajectory where $\dot{r}$ is small but not zero.

Example (Motion on a paraboloid): A particle moves under gravity on the smooth inside surface of the paraboloid $z = r^2/4a = H(r)$. Initially it is at a height $z = a$ and is projected horizontally with speed v along the surface of the paraboloid. Show that the particle moves between two heights in the subsequent motion, and find them.

Solution: At $t = 0$, $z = a$. Since $r^2 = 4az$ (the particle is on the paraboloid), initially $r = 2a$. Also

$$\left.\begin{aligned} r\dot{\theta} &= v \\ \dot{r} &= 0 \\ \dot{z} &= 0 \end{aligned}\right\}\ \text{at } t = 0.$$

where Fig. 8b in particular highlights $r\dot{\theta} = v$. Thus

$$h = r^2\dot{\theta} = r \times r\dot{\theta} = 2av. \tag{5.31}$$

Conservation of energy, on use of the initial conditions, gives

$$\frac{1}{2}m\dot{\mathbf{r}}^2 + mgz = \text{constant} = \frac{1}{2}mv^2 + mga. \tag{5.32}$$

Thus

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + gz = \frac{1}{2}v^2 + ga. \tag{5.33}$$

Eliminate $\dot{\theta}$ and r to get a first order differential equation for $z(t)$ only, as we are interested in heights of the motion.

We have

$$r = 2\sqrt{az}, \quad \dot{r} = \sqrt{\frac{a}{z}}\,\dot{z}\ . \tag{5.34}$$

Substituting into (5.32) using $\dot{\theta} = h/r^2$, and eliminating r for z, gives

$$\frac{1}{2}\left[\left(1+\frac{a}{z}\right)\dot{z}^2 + \frac{4a^2v^2}{4az}\right] + gz = \frac{1}{2}v^2 + ga\ , \tag{5.35}$$

and thus

$$\frac{1}{2}\left(1+\frac{a}{z}\right)\dot{z}^2 = \frac{1}{2}v^2\left(1-\frac{a}{z}\right) + g(a-z) = \frac{g}{z}(z-a)\left(\frac{v^2}{2g}-z\right)\ . \tag{5.36}$$

Since $z > 0$ and $\dot{z}^2 \geq 0$ it follows that

$$\left(\frac{v^2}{2g}-z\right)(z-a) \geq 0\ . \tag{5.37}$$

Therefore the particle always stays between the two heights $z = a$ and $z = v^2/2g$, at which $\dot{z} = 0$.

In particular the particle is confined to $z \geq a$ if $v^2 > 2ga$, or to $z \leq a$ if $v^2 < 2ga$, or to the horizontal circle $z = a$ if $v^2 = 2ga$. ■

6 The Kepler problem

In this section we introduce Newton's *law of universal gravitation*. This is described by an *inverse square law force*, and we show that a particle acted on by such a force moves on a conic section. This was famously first shown by Newton in his *Principia*. We also derive Kepler's laws of planetary motion, and comment very briefly on the inverse square law force of electrostatics.

6.1 Inverse square law forces and potentials

In sections 4.2 and 4.3 we introduced the notions of *conservative forces* and *central forces*. These lead to a conserved energy and conserved angular momentum, respectively. In this section we combine the two. Specifically, we are interested in forces given by the following:

Proposition Denote $r = |\mathbf{r}|$ and $\mathbf{e}_r = \mathbf{r}/r = \hat{\mathbf{r}}$ a unit vector in the direction of $\mathbf{r}$, where the latter is the position vector of a particle. Then forces of the form

$$\mathbf{F} = F(r)\,\mathbf{e}_r\ , \tag{6.1}$$

are *conservative central forces*, where the potential $V = V(r)$ depends only on the distance r to the origin.

Proof: It is immediate that (6.1) is a central force. Let $V(r)$ be a function of $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, where (x_1, x_2, x_3) are Cartesian coordinates. Then

$$-\nabla V \;=\; -\left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3}\right) \;=\; -\frac{\mathrm{d}V}{\mathrm{d}r}\left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}\right) \;=\; -\frac{\mathrm{d}V}{\mathrm{d}r}\,\mathbf{e}_r \;. \tag{6.2}$$

Then we have $\mathbf{F}$ is conservative by setting $V(r) = -\int_{r_0}^{r} F(s)\,\mathrm{d}s$ so that

$$F(r) \;=\; -\frac{\mathrm{d}V}{\mathrm{d}r} \;. \tag{6.3}$$

More specifically, for the remainder of this section we are interested in the following important example:

Definition The *inverse square law force* is a conservative central force with

$$V(r) \;=\; -\frac{\kappa}{r}, \qquad\qquad \mathbf{F} \;=\; -\frac{\kappa}{r^2}\,\mathbf{e}_r \;, \tag{6.4}$$

where κ is constant, and we have used (6.1) and (6.3) to relate the potential to the force.

Newton's law of universal gravitation

The gravitational force on a point particle at position $\mathbf{r}_1$ due to a point particle at position $\mathbf{r}_2$ is given by

$$\mathbf{F} \;=\; \mathbf{F}_{12} \;=\; -G_N \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \;=\; -G_N \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\hat{\mathbf{r}}_{12} \;. \tag{6.5}$$

Here m_1, m_2 are the (gravitational) masses of the two particles, we have defined the unit vector $\hat{\mathbf{r}}_{12} = (\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|$, and $G_N \simeq 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ is *Newton's gravitational constant.*

Note

- We have $\mathbf{F}_{12} = -\mathbf{F}_{21}$. This is Newton's third law.
- If say particle 2 has much more mass, so that $m_2 \gg m_1$, then particle 2's trajectory can be approximated as essentially stationary.
- This will be investigated in detail later, in the dynamics of systems of particles.
- We use this assumption regularly. We have already implicitly assumed the Earth is not moved by a projectile and, later, we will also assume the Sun is not moved by the Earth.
- Placing the heavy mass at the origin, so that $\mathbf{r}_2 = \mathbf{0}$ and with the relabelling $\mathbf{r} = \mathbf{r}_1$, $m_2 = M$, $m_1 = m$ Eqn (6.5) collapses to the respective force and potential energy

  $$\mathbf{F} \;=\; -\frac{\kappa}{r^2}\,\mathbf{e}_r \;, \qquad V(r) = -\frac{\kappa}{r}$$

  with $\kappa = G_N m M$, where for planetary examples M is usually the mass of the Sun.

- As the force is conservative, energy is conserved.
- **Aside** The force and potential are singular at the origin; the point particle assumption breaks down as the Sun, Earth and other bodies within the system are not point particles and cannot occupy the same point in space. We exclude $r = 0$ from our considerations.

Remark: We now apparently have *two* different descriptions of the force of gravity: one given by Newton's inverse square law force, and the other given by $\mathbf{F} = -mg\,\mathbf{k}$.

The latter is valid on the Earth's surface as an approximation to Newton's inverse square law force in the limit that the lengthscales of the dynamics are much smaller than the lengthscales over which the Earth's gravitational field varies, that is the Earth's radius. Hence there is no variation in $\boldsymbol{g} = -g\mathbf{k}$ in the approximation.

Aside. Coulomb's law of electrostatics.

Coulomb discovered a similar inverse square law force between two point charges at rest. Given two such charges q_1, q_2 at positions $\mathbf{r}_1$, $\mathbf{r}_2$, respectively, the first charge experiences an *electrostatic* force $\mathbf{F}_{12}$ due to the second charge, given by

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\hat{\mathbf{r}}_{12} . \tag{6.6}$$

The constant $\epsilon_0 \simeq 8.85 \times 10^{-12}\ \mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}$ is called the *permittivity of free space*. Unlike gravity, the Coulomb force can be both attractive and repulsive, with opposite sign charges attracting, and same sign charges repelling.

As we did for gravity, let us now suppose the second charge has much more mass and then can be approximated as fixed at the origin ($\mathbf{r}_2 = \mathbf{0}$). With the relabelling $q_2 = Q$, $\mathbf{r}_1 = \mathbf{r}$ and $q_1 = q$ we may restate Coulomb's law as:

> A point charge Q at the origin O exerts an electrostatic force $\mathbf{F}$ on a point charge q at position $\mathbf{r}$ given by (6.4), where $\kappa = -Qq/4\pi\epsilon_0$.

6.2 The Kepler problem and planetary orbits

We now consider motion due to a conservative central force to enable an investigation of planetary motion and Kepler's laws.

Exercise Determine the equations of motion for a point particle of mass m in the conservative central force (6.1),

$$\mathbf{F} = F(r)\,\mathbf{e}_r . \tag{6.7}$$

Solution From Section 4.3, we know that the motion of the particle lies in a plane, which without loss is the x-y plane, and we use plane polars r, θ below in this plane.

We already know from Section 4.3 that $mr^2\dot{\theta}$, the z-component of the angular momentum about the origin, $\mathbf{L} = \mathbf{L}_O$, is conserved.

Newton's second law is

$$m\ddot{\mathbf{r}} \;=\; F(r)\,\mathbf{e}_r \tag{6.8}$$

and thus

$$m\left[(\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\,\mathbf{e}_\theta\right] \;=\; F(r)\,\mathbf{e}_r\ . \tag{6.9}$$

Resolving in the $\mathbf{e}_\theta$, $\mathbf{e}_r$ directions, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) \;=\; 0\ , \tag{6.10}$$

$$m(\ddot{r} - r\dot{\theta}^2) \;=\; F(r), \tag{6.11}$$

and the first equation confirms $mr^2\dot{\theta} = mh$ is conserved, as expected and discussed in Section 4.3. Eliminating $\dot{\theta}$ in terms of h gives

$$m\left(\ddot{r} - \frac{h^2}{r^3}\right) \;=\; F(r)\ . \tag{6.12}$$

Solving this gives $r(t)$, and we then have a first order ODE for $\theta(t)$ via $\dot{\theta} = h/r^2(t)$. This generates the trajectory, parameterised by time.

Here, it will in fact be easier to solve for the trajectory parameterised by θ, so we look instead to solve for $r(\theta)$ directly.

It is much easier in practice with central force problems to work with

$$u(\theta) \equiv \frac{1}{r(\theta)}.$$

Proposition For a particle moving in a central force the equations of motion imply that, for $h \neq 0$,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u \;=\; -\frac{F(1/u)}{mh^2u^2}\ , \tag{6.13}$$

where $u(\theta) = 1/r(\theta)$ gives the curve traced out by the path of the particle.

Proof: We have $\dot{\theta} = h/r^2 = hu^2$, giving

$$\dot{r} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{u}\right) \;=\; -\frac{1}{u^2}\,\dot{\theta}\,\frac{\mathrm{d}u}{\mathrm{d}\theta} \;=\; -h\frac{\mathrm{d}u}{\mathrm{d}\theta}\ . \tag{6.14}$$

Differentiating again:

$$\ddot{r} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(-h\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = -h\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) \;=\; -h\dot{\theta}\,\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} \;=\; -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\ . \tag{6.15}$$

Substituting this into (6.12) gives

$$m\left(-h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - h^2u^3\right) \;=\; F\left(\frac{1}{u}\right)\ , \tag{6.16}$$

which rearranges to (6.13). ■

Aside Eqn (6.13) is not valid when $h = r^2\dot{\theta} = 0$. Since $r = 0$ is excluded, we have $\dot{\theta} = 0$ and straight line motion with θ constant. Then the parametrization $r = r(\theta)$ does not make sense. Below in this section, we assume $r^2\dot{\theta} \neq 0$ and thus, by e.g. Eqn (4.20), $\mathbf{L}_O \neq \mathbf{0}$.

The Kepler Problem We now examine the central inverse square law force for a particle of mass m, with $F(r) = -\kappa/r^2$, where $\kappa = G_N Mm > 0$; this is the *Kepler problem.*

Theorem (Due to Newton) For the Kepler problem the particle trajectories with non-zero angular momentum are conic sections.

Proof: In terms of the variable $u = 1/r$ we have $F(r) = -\kappa u^2$. Substituting this into (6.13) gives

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u \;=\; \frac{\kappa}{mh^2} \; . \tag{6.17}$$

Remarkably, the change of variable has reduced the problem to the same ODE we found for a particle attached to a linear Hooke law spring (*c.f.* equation (2.16)) which exhibits simple harmonic motion.

The general solution for $u(\theta)$ is

$$u(\theta) \;=\; \frac{\kappa}{mh^2}\left[1 + e\,\cos\left(\theta + \phi\right)\right] \; , \tag{6.18}$$

where e and ϕ are integration constants.

We can use the freedom to rotate the coordinate system axes to set either $\phi = 0$ if $e > 0$, or $\phi = \pi$ if $e < 0$ to obtain $e\cos(\theta + \pi) = -e\cos\theta = |e|\cos\theta$; hence without loss we take $e \geq 0$ and $\phi = 0$.

On the other hand, from the Prelims Geometry course we know that the general polar form of a conic may be written as

$$\frac{r_0}{r} \;=\; r_0 u \;=\; 1 + e\,\cos\theta \; , \tag{6.19}$$

where r_0 is a constant and the origin at $r = 0$ is situated at one of the foci. Comparing to (6.18) and recalling that $\kappa > 0$ we may thus identify

$$r(\theta) \;=\; \frac{r_0}{1 + e\cos\theta} \; , \qquad \text{where} \qquad r_0 \;=\; \frac{mh^2}{\kappa} \;=\; \frac{h^2}{GM} \;>\; 0 \; . \tag{6.20}$$

Regarding GM as fixed, the scale parameter r_0 is thus determined by the specific angular momentum h.

Note The integration constant $e \geq 0$ is *the eccentricity of the conic.* This is

- an ellipse for $0 \leq e < 1$, with $e = 0$ being a circle,
- a parabola for $e = 1$,
- a hyperbola for $e > 1$. ■

Time dependence The time dependence may be recovered by solving $\dot{\theta} = h\, u^2(\theta)$ as

$$h\, t \;=\; \int \frac{\mathrm{d}\theta}{u^2(\theta)} \;=\; r_0^2 \int \frac{\mathrm{d}\theta}{(1 + e\,\cos\theta)^2}\ , \tag{6.21}$$

which gives t as a function of θ.

Conics

We now show that the solution to the Kepler problem reduces to the normal form for a conic. We begin by expressing the polar form of a conic (6.20) in Cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$, whereby

$$r_0 \;=\; e\, r\cos\theta + r \;=\; e\, x + r, \quad \text{hence} \quad r \;=\; r_0 - e\, x\ . \tag{6.22}$$

Squaring both sides then gives

$$x^2 + y^2 \;=\; (r_0 - e\, x)^2\ . \tag{6.23}$$

Rearranging yields

$$x^2 + y^2 = r_0^2 - 2er_0x + e^2x^2,$$

and hence

$$(1 - e^2)\left[x^2 + \frac{2er_0}{1-e^2}\, x\right] + y^2 = r_0^2.$$

With

$$x_0 \;=\; -\frac{e\, r_0}{1-e^2}, \quad k = r_0/e$$

completing the square gives

$$(1-e^2)(x-x_0)^2 + y^2 = r_0^2 + \frac{e^2r_0^2}{1-e^2} = \frac{r_0^2}{1-e^2} = \frac{e^2k^2}{1-e^2},$$

which is the normal form for a conic section, with a shifted origin.

Its collapse onto an ellipse, hyperbola or parabola according to the value of e is lectured in detail in Prelims Geometry. For completeness, the reduction is briefly reproduced below, but this will not be lectured.

Ellipses: $0 \le e < 1$: In this case we define

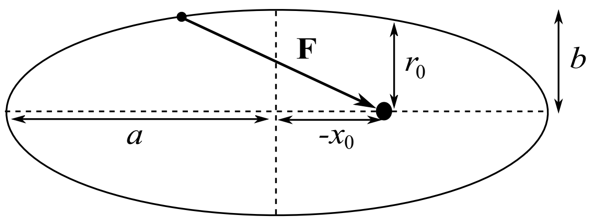
$$a = \frac{r_0}{(1-e^2)}\ , \quad b \;=\; \frac{r_0}{(1-e^2)^{1/2}}\ , \quad x_0 \;=\; -\frac{e\, r_0}{1-e^2} \;=\; -e\, a \ < 0, \tag{6.24}$$

Completing the square for (6.23) generates

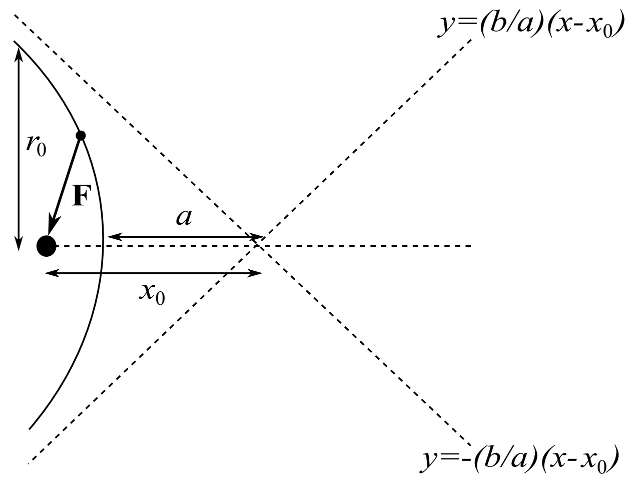
$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} \;=\; 1\ , \tag{6.25}$$

which is plotted in Figure 13a.

This is the equation of an ellipse centred at $(x_0, 0)$, with a semi-major axis of length a and a semi-minor axis of length $b \le a$. One of the two foci is located at the origin $(x, y) = (0, 0)$, the centre of attraction $r = 0$ for the inverse square law force.



(a) An ellipse. The large black dot is the origin, which is one of the foci and also the centre of attraction of the inverse square law force. The centre of the ellipse is $(x_0, 0)$, where $x_0 = -e\,a \leq 0$. The semi-major axis has length a, while the semi-minor axis has length $b \leq a$.



(b) A hyperbola. The large black dot is again the origin, focus, and centre of the force. The two asymptotes are $y = \pm(b/a)(x - x_0)$, which meet at the point $(x_0, 0)$, where now $x_0 = e\,a > 0$.

Figure 13: Conic sections.

When $e = 0$, we have $a = b = r_0$, $x_0 = 0$ and the ellipse is a circle centred at the origin.

Hyperbolae: $e > 1$: In this case we similarly define

$$a = \frac{r_0}{(e^2 - 1)} , \quad b = \frac{r_0}{(e^2 - 1)^{1/2}} , \quad x_0 = \frac{e\,r_0}{e^2 - 1} = e\,a > 0. \tag{6.26}$$

Some algebra reveals that (6.23) reduces to

$$\frac{(x - x_0)^2}{a^2} - \frac{y^2}{b^2} = 1 . \tag{6.27}$$

This is the equation of a hyperbola, plotted in Figure 13b. .

There are two asymptotes $y = \pm(b/a)(x - x_0)$, dropping the "1" from the right hand side of (6.27), which meet at $x = x_0$.

The focus is at the origin $(x, y) = (0, 0)$, which is again the centre of the inverse square law force.

Notice from (6.20) that $r \to \infty$ along the asymptotes for $\cos\theta = -1/e$, which has two solutions $\theta = \pm\theta_0$, where $\theta_0 = \cos^{-1}(-1/e) > \pi/2$ and θ is the angle subtended at the origin.

Parabolae: $e = 1$: Equation (6.23) reads simply

$$y^2 = r_0^2 - 2r_0 x , \tag{6.28}$$

which is the equation of a parabola. This is again an unbounded orbit, where now $r \to \infty$ for $\cos\theta = -1$, *i.e.* $\theta = \pm\pi$.

The effective potential and energy

We now reconsider the original equation of motion (6.12) for $r(t)$ from a context of energy. Recalling that $F(r) = -\mathrm{d}V/\mathrm{d}r$ and defining an effective potential

$$V_{\text{eff}}(r) \;=\; V(r) + \frac{mh^2}{2r^2}, \tag{6.29}$$

we have (6.12) may be written in the form

$$m\ddot{r} \;=\; -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} \;. \tag{6.30}$$

The equation of motion (6.30) now has the structure of motion in one dimension, with an effective potential energy V_{eff}.

Furthermore, as may be deduced from the original equation of motion (6.12) via the techniques of Section (3.1), the energy

$$\begin{aligned} E \;&=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(r) \;=\; \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \\ &=\; \frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) + V(r) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) \end{aligned} \tag{6.31}$$

is conserved, where $h = r^2\dot{\theta}$ has been used.

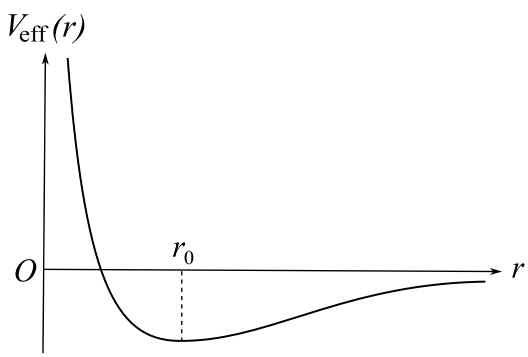


Figure 14: The effective potential $V_{\text{eff}}(r)$ for the Kepler inverse square law force problem, where V_{eff} has a unique local minimum at $r = r_0$.

For the Kepler problem we have $V(r) = -\kappa/r$, the effective potential is shown in Figure 14.

A solution with $r = r_0$ constant has $\ddot{r} = 0$, and thus from (6.30) r_0 is a turning point of the effective potential.

One may easily check that

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}(r_0) \;=\; 0 \qquad \Longrightarrow \qquad r_0 \;=\; \frac{mh^2}{\kappa} \;. \tag{6.32}$$

Of course the circular trajectory $r = r_0$ constant is consistent with the general solution (6.20) with eccentricity $e = 0$.

Being a local minimum of the effective potential also means that this circular orbit is *stable* to small perturbations of r, as we learned in section 3.3.

The Energy, in terms of the eccentricity, angular momentum and physical parameters

We already have the total energy, E in Eqn. (6.31), is conserved. To simplify this expression, given the solution of the equation of motion (6.18), we need $\dot{r}$, which from Eqns. (6.14), (6.18), is given by

$$\dot{r} \;=\; -h\,\frac{\mathrm{d}u}{\mathrm{d}\theta} \;=\; \frac{he}{r_0}\sin\theta\ . \tag{6.33}$$

Inserting this and $r = r_0/(1+e\,\cos\theta)$ from Eqn (6.20) into

$$E \;=\; \frac{1}{2}m\dot{r}^2 - \frac{\kappa}{r} + \frac{mh^2}{2r^2}\ , \tag{6.34}$$

and, using $r_0 = mh^2/\kappa$ from Eqn. (6.32), we find that E is indeed constant:

$$E \;=\; \frac{(e^2-1)\kappa^2}{2mh^2}\ . \tag{6.35}$$

■

In particular we see that the bound orbits with $0 \le e < 1$ (*i.e.* ellipses) have $E < 0$.

This is also clear from the effective potential in Figure 14: for $E < 0$ the particle moves back and forth between some r_{min} and r_{max}, and the orbit is bounded, as in the discussion of a general potential in section 3.2.

On the other hand for $e > 1$ we have $E > 0$ and the particle has a minimum radius, but escapes to infinity. These are the hyperbolic orbits.

The parabola $e = 1$ is the limiting case with zero energy, for which the particle only just escapes to infinity, where the potential for the inverse square law is zero.

6.2.1 Examples

Example (Geostationary orbit): A *geostationary orbit* is a circular orbit in the plane containing the Earth's equator, which co-rotates with the Earth. Determine the altitude of a satellite of mass m on this geostationary orbit.

Solution The angular velocity of the satellite on the geostationary orbit is the same as that of the Earth's rotation, namely $\dot{\theta} = 2\pi$ radians per day.

Using $h = r_0^2\,\dot{\theta}$, with $\kappa = G_N M_E\, m$, where M_E is the mass of the Earth, and m the mass of the satellite, Eqn. (6.32) implies the radius satisfies

$$r_0 \;=\; \frac{mh^2}{\kappa} \;=\; \frac{r_0^4\,\dot{\theta}^2}{G_N M_E}$$

and hence

$$r_0 \;=\; \left(\frac{G_N M_E}{\dot{\theta}^2}\right)^{1/3} \;\simeq\; 4.22\times 10^7\,\mathrm{m} \;=\; 42,200\ \mathrm{km}\ , \tag{6.36}$$

using $G_N \simeq 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$, $M_E \simeq 5.97 \times 10^{24}$ kg, $\dot{\theta} \simeq 7.27 \times 10^{-5}\ \mathrm{s}^{-1}$.

Noting the radius of the Earth is $r_E \simeq 6370$km we have the altitude of the satellite is

$$r_0 - r_E \simeq 35,800\text{km}.$$

■

Example (Angle of deflection of a comet): A comet of mass m approaches the Sun, of mass M_S, from a very large distance with speed v. If the Sun exerted no force on the comet it would continue with uniform velocity on an undeflected path, giving a distance of closest approach to the Sun of p. Find the actual path of the comet and determine the angle through which it is deflected.

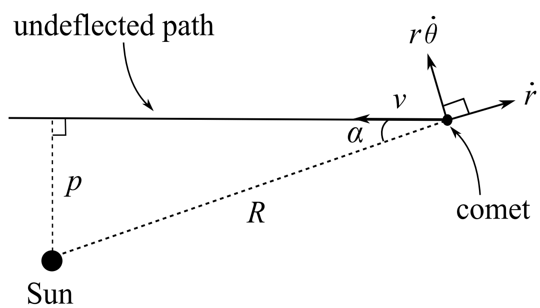


Figure 15: A comet of mass m approaching the Sun from a very large distance with speed v. Without the effect of gravity the comet travels undeflected with constant speed v, and its closest approach is the distance p. At time $t = -T$ with $T \gg p/v$ large, the particle is depicted. Here $p = R\sin\alpha$, and the angle α is very small and T-dependent, tending to zero as $T \to \infty$, while $R \to \infty$ in the same limit.

Solution: Figure 15 shows the comet's path undeflected by gravity.

Initial conditions Let T be fixed, with dimensions of time and such that $T \gg p/v$. Consider the time with $t = -T$ and let α, R be the T-dependent angle and radial distance in Figure 15, where the comet is depicted at time T.

- We have initial conditions at time $t = -T$ that

$$\dot{r} = -v\cos\alpha, \qquad r\dot{\theta} = v\sin\alpha = p\,v/R,$$

  where in the latter equation we have used $p = R\sin\alpha$.

- Note $\alpha \to 0$, $R \to \infty$ when we take the limit $T \to \infty$ below.

- The conserved specific angular momentum h may be computed from these initial conditions as

$$h \;=\; r^2\dot{\theta} \;=\; p\,v\;. \tag{6.37}$$

- Thus at time $t = -T$ we have the initial conditions

$$u = \frac{1}{R}, \qquad \frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{\dot{r}}{h} = \frac{1}{p}\cos\alpha. \tag{6.38}$$

- At time $t = -T$ we rotate the axes without loss so that the comet is at angle $\theta = 0$ at this time.

The solution of the Kepler Problem and the path of the comet

The general solution to the Kepler problem may be written

$$u(\theta) \;=\; \frac{\kappa}{mh^2} + C\,\cos\theta + D\,\sin\theta\ . \tag{6.39}$$

We use the initial conditions (6.38) at $t = -T$, $\theta = 0$, to determine $C,\ D$ and hence

$$u(\theta) \;=\; \frac{\kappa}{mp^2v^2}(1-\cos\theta) + \frac{1}{R}\cos\theta + \frac{1}{p}\cos\alpha\sin\theta\ . \tag{6.40}$$

This holds for all sufficiently large T.

A posteriori, taking the limit $T \to \infty$ after all other calculations are completed, and noting $R \to \infty$, $\alpha \to 0$ in this limit, we have the path of the comet is given by

$$u(\theta) \;=\; \frac{\kappa}{mp^2v^2}(1-\cos\theta) + \frac{1}{p}\sin\theta\ . \tag{6.41}$$

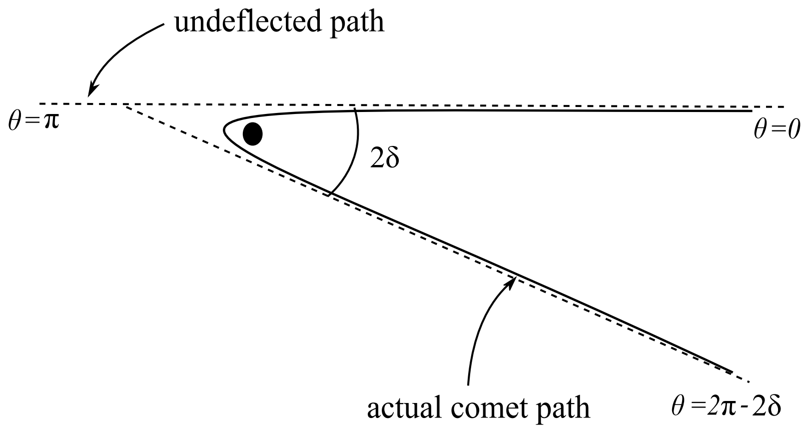


Figure 16: The actual path of the comet. The origin is at the Sun (large black dot), with the $\theta = 0$ axis horizontal, to the right (one should understand the dotted lines as extending to infinity).

The angle of deflection

The comet asymptotes to infinity at angles given by $u = 1/r = 0$. This gives

$$\frac{\kappa}{mp^2v^2}(1-\cos\theta) + \frac{1}{p}\sin\theta \;=\; 0\ . \tag{6.42}$$

Using double angle formulas we may rewrite this as

$$\frac{\kappa}{mp^2v^2}\sin^2\frac{\theta}{2} + \frac{1}{p}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \;=\; 0\ . \tag{6.43}$$

Clearly one solution is $\theta = 0$, corresponding to the $t = -T \to -\infty$ limit by construction.

We are thus interested in the other solution, which satisfies

$$\frac{\kappa}{mp^2v^2} \sin\frac{\theta}{2} + \frac{1}{p}\cos\frac{\theta}{2} \quad = \quad 0 \tag{6.44}$$

and hence

$$\tan\frac{\theta}{2} \quad = \quad -\frac{mpv^2}{\kappa} \ . \tag{6.45}$$

Noting the deflection depicted in Figure 16, we set $\theta = 2\pi - 2\delta$. Using

$$\tan(\pi - \delta) = -\tan\delta,$$

we thus have

$$\tan\delta \quad = \quad \frac{mp\,v^2}{\kappa} \quad = \quad \frac{p\,v^2}{G_N M_S} \ . \tag{6.46}$$

Here in the second equality we have inserted the value $\kappa = G_N M_S\, m$, where M_S is the mass of the Sun.

As illustrated in Figure 16 the comet comes in at an angle $\theta = 0$, goes past the Sun and proceeds to infinity at an angle $2\pi - 2\delta$. Noting that no deflection would correspond to an outward asymptote with angle π, it follows that the comet is deflected through an angle

$$(2\pi - 2\delta) - \pi = \pi - 2\delta = \pi - 2\tan^{-1}\left(\frac{p\,v^2}{G_N M_S}\right) .$$

■

6.3 Kepler's laws

In the late 16th century the Danish nobleman Tycho Brahe made accurate and comprehensive planetary observations, which Johannes Kepler was then able to analyse. Using this empirical data Kepler remarkably deduced the following three laws (published between 1609 and 1619):

> **K1**: The path of each planet is an ellipse with the Sun at the focus.
>
> **K2**: A straight line joining the Sun and a planet sweeps out equal areas in equal times.
>
> **K3**: The square of each planet's period is proportional to the cube of the semi-major axis of its elliptical orbit.

The force attracting a planet to the Sun is of course Newton's inverse square law of gravitation.

In the below, we explicitly ignore the fact that in our solar system there are many planets, which also attract each other and thus we treat each planet individually, neglecting all others. Like the other approximations we have made thus far, for instance that the Sun does not move relative to an inertial reference frame, this also has excellent accuracy.

Proof of K1 Putting the Sun at the origin, and neglecting all but of one of the planets to consider its trajectory, we thus have shown **K1** from Newton's laws already, from the solution to Kepler's problem.

6.3.1 Kepler's laws K2 and K3

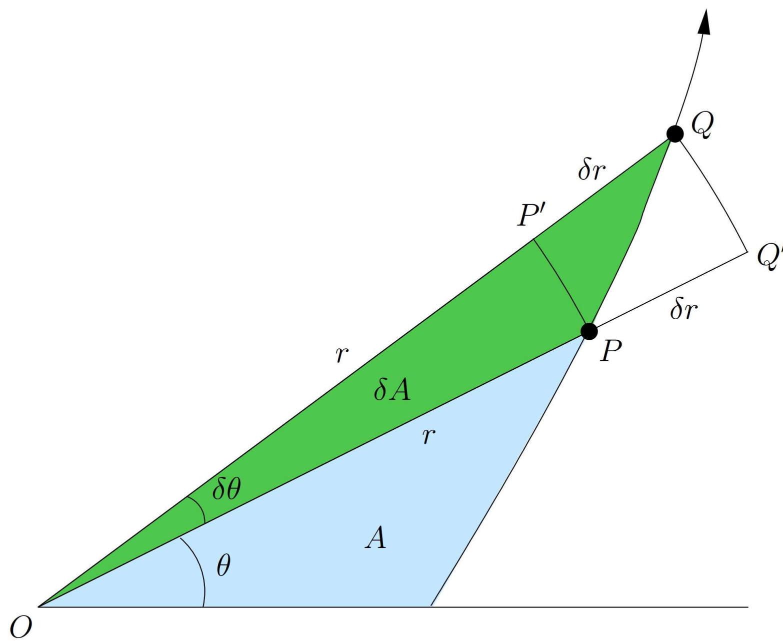


Figure 17: A particle moves from a point P at time t to a point Q at time $t + \delta t$, with the angle subtended at the origin changing by a small amount $\delta\theta$, sweeping out a region $OPQO$, of area δA as shown in green. Let r denote the distance $|OP|$, with $r + \delta r$ the distance $|OQ|$, P' the point on OQ such that $|OP'| = r$ and Q' the point on OP, such that $|OQ'| = r + \delta r$.

Proof of K2: Kepler's second law is a simple consequence of conservation of angular momentum.

Recall from conservation of angular momentum we have $r^2\dot{\theta} = h =$ constant.

A straight line from the Sun to a planet is simply the position vector $\mathbf{r}(t)$. In a small time interval δt, as shown in Figure 17) the planet sweeps out a region $OPQO$ that is approximately triangular.

In the first instance, with the points O, P, P', Q, Q' as given in Figure 17, we consider $\delta r = |OQ| - |OP| \geq 0$, with $r = |OP|$.

Noting the area of the circular sector, of radius s, subtending an angle ψ between its radii, is given by $\psi s^2/2$, we have

$$\frac{1}{2}r^2\delta\theta \leq \delta A \leq \frac{1}{2}(r + \delta r)^2\delta\theta.$$

Dividing by δt and taking $\delta t \to 0$, so that $\delta r \to 0$, $\delta\theta \to 0$, we have

$$\dot{A} \;=\; \frac{1}{2}r^2\dot{\theta} \;=\; \frac{1}{2}h \;=\; \text{constant}, \tag{6.47}$$

which also holds by an analogous argument when $\delta r < 0$. ■

Aside Being a consequence only of conservation of angular momentum, Kepler's second law holds for *any* central force (even non-conservative ones).

Proof of K3:

Recall that the area of an ellipse with semi-major axis a and semi-minor axis b is

$$A \;=\; \pi ab\;. \tag{6.48}$$

We know from **K2** that this area is swept out at a constant rate $\dot{A} = \frac{1}{2}h$. Integrating this over one period we obtain

$$A \;=\; \int \mathrm{d}A \;=\; \frac{1}{2}h\int \mathrm{d}t \;=\; \frac{1}{2}hT\;. \tag{6.49}$$

Thus the square of the period T is

$$T^2 \;=\; \frac{4A^2}{h^2} \;=\; \frac{4\pi^2 a^2 b^2}{h^2} \;=\; \frac{4\pi^2}{G_N M_S}\cdot\frac{b^2}{ar_0}\cdot a^3\;, \tag{6.50}$$

where in the last step we have substituted $h^2 = \kappa r_0/m = G_N M_S r_0$ using (6.20), where $\kappa = G_N m M_S$, m is the mass of the planet and M_S is the mass of the Sun.

Recall from Eqn (6.24) for the equation of a Kepler problem ellipse, we have

$$a = \frac{r_0}{(1-e^2)}\;,\qquad b \;=\; \frac{r_0}{(1-e^2)^{1/2}}\;, \tag{6.51}$$

and thus

$$\frac{b^2}{ar_0} = 1.$$

Hence

$$T^2 \;=\; \frac{4\pi^2}{G_N M_S}\,a^3, \tag{6.52}$$

which is precisely Kepler's third law. ■

7 Systems of particles

So far we have mainly been studying the motion of a single particle. We now proceed to consider many particles.

7.1 Centre of mass motion

Note on notation: Henceforth we will *always* denote our inertial frame, in which we write down Newton's second law, as $\hat{\mathcal{S}}$, with origin $\hat{O}$.

Consider a system of N point particles. With respect to an inertial frame $\hat{\mathcal{S}}$, we denote the position vector of the Ith particle from $\hat{O}$ by $\mathbf{r}_I$, which has mass m_I and hence linear momentum $\mathbf{p}_I = m_I\dot{\mathbf{r}}_I$, $I = 1, \ldots, N$.

We suppose that particle J exerts a force $\mathbf{F}_{IJ}$ on particle I, for $I \neq J$. Newton's third law immediately tells us that $\mathbf{F}_{JI} = -\mathbf{F}_{IJ}$ for each $I \neq J$.

Without loss set $\mathbf{F}_{II} = \mathbf{0}$.

On the other hand Newton's second law for particle I reads

$$m_I\ddot{\mathbf{r}}_I \;=\; \dot{\mathbf{p}}_I \;=\; \mathbf{F}_I \;=\; \mathbf{F}_I^{\mathrm{ext}} + \sum_{J\neq I} \mathbf{F}_{IJ} \; . \tag{7.1}$$

Here we have included an *external force* $\mathbf{F}_I^{\mathrm{ext}}$, *i.e.* a force acting on particle I that is not due to the other $N-1$ particles in the system. We refer to the $\mathbf{F}_{IJ}$ as *internal forces*.

Aside. When considering a single particle, the force $\mathbf{F} = \mathbf{F}^{\mathrm{ext}}$ in Newton's second law is by definition always external.

Definition The *centre of mass* of the system of particles is the point G, with position vector

$$\mathbf{R}_G \;\equiv\; \frac{1}{M}\sum_{I=1}^{N} m_I \mathbf{r}_I \; , \tag{7.2}$$

where $M = \sum_{I=1}^{N} m_I$ is the *total mass*. Similarly the *total momentum* of the system is

$$\mathbf{P} \;\equiv\; \sum_{I=1}^{N} \mathbf{p}_I \;=\; M\dot{\mathbf{R}}_G \; . \tag{7.3}$$

Theorem The centre of mass of the system behaves like a point particle of mass M acted on by the *total external force*. In particular, the dynamics of the centre of mass is independent of the internal forces.

Proof: We have

$$M\ddot{\mathbf{R}}_G \;=\; \dot{\mathbf{P}} \;=\; \sum_{I=1}^{N} \dot{\mathbf{p}}_I \;=\; \sum_{I=1}^{N}\Big(\mathbf{F}_I^{\mathrm{ext}} + \sum_{J\neq I}\mathbf{F}_{IJ}\Big) \; . \tag{7.4}$$

However, due to Newton's third law $\mathbf{F}_{IJ} = -\mathbf{F}_{JI}$, the $N(N-1)$ terms in the sum

$$\sum_{I=1}^{N}\sum_{J\neq I}\mathbf{F}_{IJ} \quad = \quad \mathbf{0} \tag{7.5}$$

cancel pairwise.

Thus (7.4) becomes

$$M\ddot{\mathbf{R}}_G \quad = \quad \dot{\mathbf{P}} \quad = \quad \sum_{I=1}^{N}\mathbf{F}_I^{\text{ext}} \quad = \quad \mathbf{F}^{\text{ext}} \ , \tag{7.6}$$

where $\mathbf{F}^{\text{ext}}$ is by definition the total external force. ■

This result explains why we can often so accurately model objects as point particles, even when they manifestly are not. Whatever internal forces are acting within our object, for example holding it together, they will cancel out of the centre of mass motion. In most of the problems we have studied we have really been modelling the centre of mass motion of an object, and applying Newton's second law in the form (7.6).

Definition A *closed system* is one in which all forces are internal, acting between the constituents of the system. That is, $\mathbf{F}_I^{\text{ext}} = \mathbf{0}$, $I = 1, \ldots, N$.

Hence, for a closed system the total momentum is conserved, $\dot{\mathbf{P}} = \mathbf{0}$. and hence the centre of mass moves with constant velocity $\dot{\mathbf{R}}_G =$ constant. Without loss, we may then take the centre of mass to be $\mathbf{R}_G = \mathbf{0}$, the origin of our inertial reference frame.

Definition *A centre of mass reference frame* has its origin at the centre of mass, $\mathbf{R}_G = \mathbf{0}$. When $\mathbf{F}^{\text{ext}} = \mathbf{0}$, this is also an inertial reference frame.

Definition The *total angular momentum* $\mathbf{L} = \mathbf{L}_P$ of the system about a point P is

$$\mathbf{L}_P \quad = \quad \sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x}) \wedge \mathbf{p}_I \ , \tag{7.7}$$

where P has position vector $\mathbf{x}$ from the origin $\hat{O}$. That is, $\mathbf{L}$ is the vector sum of the angular momenta $\mathbf{L}_I = (\mathbf{r}_I - \mathbf{x}) \wedge \mathbf{p}_I$ for each particle I about P.

As noted previously, Eqn. (4.18), $p_I = m_I\dot{\mathbf{r}}_I$ is the velocity of the particle in the inertial frame, *not* the velocity relative to P, which in general may be moving, $\mathbf{x} = \mathbf{x}(t)$.

Using the definition (7.7) we begin by computing

$$\begin{aligned}
\dot{\mathbf{L}}_P &= \sum_{I=1}^{N}[(\dot{\mathbf{r}}_I - \dot{\mathbf{x}}) \wedge \mathbf{p}_I + (\mathbf{r}_I - \mathbf{x}) \wedge \dot{\mathbf{p}}_I] \\
&= -\dot{\mathbf{x}} \wedge \mathbf{P} + \sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x}) \wedge \dot{\mathbf{p}}_I \\
&= -\dot{\mathbf{x}} \wedge \mathbf{P} + \sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x}) \wedge \left(\mathbf{F}_I^{\rm ext} + \sum_{J\neq I}\mathbf{F}_{IJ}\right) . \qquad (7.8)
\end{aligned}$$

Here in the second equality we have used $\dot{\mathbf{r}}_I \wedge \mathbf{p}_I = \dot{\mathbf{r}}_I \wedge m_I\dot{\mathbf{r}}_I = \mathbf{0}$.

The third equality uses Newton's second law (7.1).

In $\sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x}) \wedge \sum_{J\neq I}\mathbf{F}_{IJ}$ we again have $\frac{1}{2}N(N-1)$ pairs of terms, which look like

$$(\mathbf{r}_I - \mathbf{x}) \wedge \mathbf{F}_{IJ} + (\mathbf{r}_J - \mathbf{x}) \wedge \mathbf{F}_{JI} \;=\; (\mathbf{r}_I - \mathbf{r}_J) \wedge \mathbf{F}_{IJ} \ , \qquad (7.9)$$

and we have used Newton's third law. To get any further we need the *strong form* of Newton's third law:

> **N3** (strong form): If particle 1 exerts a force $\mathbf{F} = \mathbf{F}_{21}$ on particle 2, then particle 2 also exerts a force $\mathbf{F}_{12} = -\mathbf{F}$ on particle 1. *In addition,* this force acts along the vector connecting particle 1 to particle 2, $\mathbf{F}_{12} \propto (\mathbf{r}_1 - \mathbf{r}_2)$.

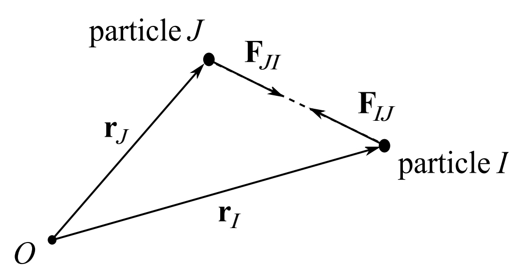


Figure 18: The strong form of Newton's third law.

This clearly holds for the inverse square law forces of Newton (6.5) and Coulomb (6.6), but there are examples that don't satisfy this.[3]

Returning to (7.9), we see that if the strong form of Newton's third law holds this is zero, and hence (7.8) gives

$$\dot{\mathbf{L}}_P \;=\; -\dot{\mathbf{x}} \wedge \mathbf{P} + \sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x}) \wedge \mathbf{F}_I^{\rm ext} \;=\; -\dot{\mathbf{x}} \wedge \mathbf{P} + \boldsymbol{\tau}_P^{\rm ext} \ , \qquad (7.10)$$

[3]Notably the magnetic component of the Lorentz force.

where $\boldsymbol{\tau}_P^{\rm ext}$ is by definition the *total external torque* about P, *c.f.* (4.22).

There are two special cases where the first term on the right hand side of (7.10) is zero:

(i) taking $P = \hat{O}$, the inertial frame origin, giving $\mathbf{x} = \mathbf{0}$,

(ii) taking instead $P = G$ we have $\dot{\mathbf{x}} \wedge \mathbf{P} = \dot{\mathbf{R}}_G \wedge \mathbf{P} = \dot{\mathbf{R}}_G \wedge M\dot{\mathbf{R}}_G = \mathbf{0}$.

We have thus proven:

Theorem Provided the strong form of Newton's third law holds, the rate of change of total angular momentum about either the inertial reference frame origin, $\hat{O}$, or the centre of mass, G, equals the total external torque about the same respective point. That is,

$$\dot{\mathbf{L}}_{\hat{O}} = \boldsymbol{\tau}_{\hat{O}}^{\rm ext} \ , \qquad \dot{\mathbf{L}}_G = \boldsymbol{\tau}_G^{\rm ext} \ . \tag{7.11}$$

Corollary For a closed system satisfying the strong form of Newton's third law, the total angular momentum about the origin of an inertial frame, or about the centre of mass, is conserved.

The main application of (7.11) will be to rigid body motion, which is considered later below. In particular the following result will be useful:

Proposition Consider the system of particles in a uniform gravitational field, with acceleration due to gravity $-g\mathbf{k}$. Assuming this is the only external force acting, the total external torque about a point P with position vector $\mathbf{x}$ is

$$\boldsymbol{\tau}_P^{\rm ext} \ = \ -(\mathbf{R}_G - \mathbf{x}) \wedge Mg\,\mathbf{k} \ . \tag{7.12}$$

This is the same as the torque for a particle of mass M at the centre of mass $\mathbf{R}_G$ (compare to (4.22)). In particular, the torque about G (for which $\mathbf{x} = \mathbf{R}_G$) is zero.

Proof: We have

$$\boldsymbol{\tau}_P^{\rm ext} \ \equiv \ \sum_{I=1}^N (\mathbf{r}_I - \mathbf{x}) \wedge \mathbf{F}_I^{\rm ext} \ = \ \sum_{I=1}^N (\mathbf{r}_I - \mathbf{x}) \wedge (-m_I g\,\mathbf{k}) \ = \ -(\mathbf{R}_G - \mathbf{x}) \wedge Mg\,\mathbf{k} \ ,$$

where we have used the definitions $M = \sum_{I=1}^N m_I$, $M\mathbf{R}_G = \sum_{I=1}^N m_I \mathbf{r}_I$ in the final equality. ■

7.2 The two-body problem

The *two-body problem* is a closed system of two point particles. Newton's second and third laws give

$$m_1 \ddot{\mathbf{r}}_1 \ = \ \mathbf{F}_{12} \ , \qquad m_2 \ddot{\mathbf{r}}_2 \ = \ \mathbf{F}_{21} \ = \ -\mathbf{F}_{12} \ . \tag{7.13}$$

Adding these two equations implies that the centre of mass

$$\mathbf{R}_G \ = \ \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \tag{7.14}$$

moves with constant velocity, as also deduced in the last subsection.

On the other hand, if we define $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ so that

$$\mathbf{r}_1 \;=\; \mathbf{R}_G + \frac{m_2}{m_1+m_2}\mathbf{r}\ , \qquad \mathbf{r}_2 \;=\; \mathbf{R}_G - \frac{m_1}{m_1+m_2}\mathbf{r}\ , \tag{7.15}$$

then from (7.13) we deduce

$$\ddot{\mathbf{r}} \;=\; \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 \;=\; \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_{12} \;=\; \frac{m_1+m_2}{m_1 m_2}\mathbf{F}_{12}\ . \tag{7.16}$$

Definition The *reduced mass* for the two-body problem is $\mu = \dfrac{m_1 m_2}{m_1+m_2}$.

In terms of this the equation of motion (7.16) reads

$$\mu\ddot{\mathbf{r}} \;=\; \mathbf{F}_{12}\ . \tag{7.17}$$

Example: For the inverse square law force, with $r = |\mathbf{r}|$, we have

$$\mathbf{F}_{12} = -\frac{\kappa}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2) \;=\; -\frac{\kappa}{r^2}\frac{\mathbf{r}}{r}.$$

We have thus *effectively* reduced the two-body problem to a problem for a *single* particle, with position vector $\mathbf{r}(t)$ satisfying (7.17). The force on the right hand side is then *effectively* an external force.

Having solved this, the solution to the original two-body problem is given by (7.15).

We may thus view what we did in solving the Kepler problem in Section 6.2 in two different ways:

- If we take the mass $m = \mu$ in (6.8), then in section 6.2 we were really solving (7.17) for the two-body problem. This describes the exact internal relative motion of the two bodies.
- Suppose instead we focus on the case mass $m_1 \ll m_2$, as we implicitly did in Section 6.2 on Kepler's problem and planetary orbits.

  Without loss, $\mathbf{R}_G = \mathbf{0}$ since the centre of mass reference frame is also an inertial reference frame as there is no external force. Hence

  $$\mu\ddot{\mathbf{r}} = -\frac{\kappa}{r^2}\frac{\mathbf{r}}{r}, \qquad \mu = m_1\left(\frac{1}{1+\frac{m_1}{m_2}}\right) \simeq m_1,$$
  $$\mathbf{r}_1 = \left(\frac{1}{1+\frac{m_1}{m_2}}\right)\mathbf{r} \simeq \mathbf{r}, \qquad \mathbf{r}_2 = -\left(\frac{\frac{m_1}{m_2}}{1+\frac{m_1}{m_2}}\right)\mathbf{r} \simeq \mathbf{0}.$$

  Once m_1/m_2 is sufficiently small, we thus have that the solution in Section 6.2, with $m = m_1$ the smaller mass and the larger mass at the origin, accurately approximates the two body solution which accommodates Newton's third law and the motion of the larger body.

What is remarkable about the two-body problem is that the exact solution and approximate solution we have described are mathematically equivalent, differing only in which mass to use in Newton's second law.

8 Rotating frames and rigid bodies

In this final section we discuss two topics that involve rotation: the dynamics of *rigid bodies* in sections 8.2 and 8.3, and Newton's laws in a rotating (*i.e.* non-inertial) frame from section 8.4.

8.1 Rotating frames

Throughout this section there will always be *two* reference frames:

- A fixed reference inertial frame $\hat{\mathcal{S}}$: this has origin $\hat{O}$ and fixed coordinate axes with corresponding right handed Cartesian basis vectors $\hat{\mathbf{e}}_i$, $i = 1, 2, 3$.
- A general reference frame $\mathcal{S}$, with origin O at position vector $\mathbf{x}$ relative to $\hat{O}$, and right handed Cartesian coordinate axes associated with the basis vectors $\mathbf{e}_i$, $i = 1, 2, 3$. Typically in these sections $\mathbf{e}_i$, $i = 1, 2, 3$ will be rotating relative to the axes of $\hat{\mathcal{S}}$.

Newton's laws of motion are written in the inertial reference frame $\hat{\mathcal{S}}$, also sometimes referred as the *laboratory frame*, though the laws of motion may then be subsequently modified, so as to be relative to the axes of the non-inertial reference frame $\mathcal{S}$.

Without loss we can take the origin of inertial reference frame, $\hat{O}$, to be *fixed*. The Cartesian axes are also *time-independent* so that the Cartesian basis vectors $\hat{\mathbf{e}}_i$ are independent of time,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{e}}_i = \mathbf{0}, \qquad i = 1, 2, 3.$$

For example, when rigid bodies are introduced, the frame $\mathcal{S}$ will be chosen to rotate with the body, and thus the Cartesian basis vectors $\{\mathbf{e}_i\}$ will rotate relative to the inertial reference frame, with non-zero time derivatives.

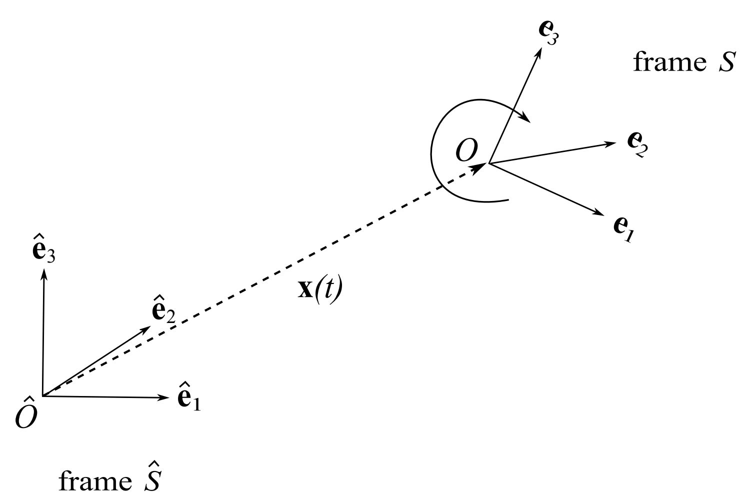


Figure 19: The reference frame $\hat{\mathcal{S}}$ is a fixed inertial reference frame, and Newton's laws of motion are written in this reference frame. With respect to $\hat{\mathcal{S}}$, a general reference frame $\mathcal{S}$ has origin O at position vector $\mathbf{x} = \mathbf{x}(t)$ as measured from $\hat{O}$, and its coordinate axes may be rotating, so that $\mathbf{e}_i = \mathbf{e}_i(t)$.

Thus, we may write the Cartesian orthonormal basis vectors $\{\mathbf{e}_i\}$ of the frame $\mathcal{S}$ as

$$\mathbf{e}_i(t) \quad = \quad \sum_{j=1}^{3} \mathcal{R}_{ij}(t)\, \hat{\mathbf{e}}_j \ , \qquad i = 1, 2, 3 \ . \tag{8.1}$$

As you learned in the Geometry course, the fact that both bases are *orthonormal and right handed* means that $\mathcal{R} = (\mathcal{R}_{ij})$ is an *orthogonal rotation matrix*. The main result of this subsection is:

Proposition There is a (unique) vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ such that

$$\dot{\mathbf{e}}_i \quad = \quad \boldsymbol{\omega} \wedge \mathbf{e}_i \ , \qquad i = 1, 2, 3 \ . \tag{8.2}$$

$\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ is called the *angular velocity* of the frame $\mathcal{S}$ with respect to fixed inertial frame $\hat{\mathcal{S}}$.

This result is lectured in extensive detail in Section 5, Video 11 of Prelims Geometry.

Proof: Take the time derivative of (8.1) to find

$$\dot{\mathbf{e}}_i \quad = \quad \sum_{j=1}^{3} \dot{\mathcal{R}}_{ij}\, \hat{\mathbf{e}}_j \quad = \quad \sum_{j,k=1}^{3} \dot{\mathcal{R}}_{ij}\mathcal{R}_{kj}\, \mathbf{e}_k \quad = \quad \sum_{k=1}^{3} (\dot{\mathcal{R}}\,\mathcal{R}^{\mathrm{T}})_{ik}\, \mathbf{e}_k \ , \tag{8.3}$$

where in the second equality we have used the fact that $\mathcal{R}$ is orthogonal, and hence $\mathcal{R}^{-1} = \mathcal{R}^{\mathrm{T}}$. Noting $\mathcal{R}\mathcal{R}^T = \mathcal{I}$, the identity, we have $\dot{\mathcal{R}}\mathcal{R}^T + \mathcal{R}\dot{\mathcal{R}}^T = \mathbf{0}$, and hence

$$\dot{\mathcal{R}}\mathcal{R}^T = -\mathcal{R}\dot{\mathcal{R}}^T = -(\dot{\mathcal{R}}\mathcal{R}^T)^T.$$

Hence $(\dot{\mathcal{R}}\,\mathcal{R}^{\mathrm{T}})$ is an anti-symmetric matrix, so we can write

$$\dot{\mathcal{R}}\,\mathcal{R}^{\mathrm{T}} \quad = \quad \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \ . \tag{8.4}$$

Then (8.3) is equivalent to (8.2) with $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ as may be confirmed by comparing the expressions after expanding. For example both give

$$\dot{\mathbf{e}}_1 = \omega_3 \mathbf{e}_2 - \omega_2 \mathbf{e}_3.$$

■

This formula, $\dot{\mathbf{e}}_i = \boldsymbol{\omega} \wedge \mathbf{e}_i$ $(i = 1, 2, 3)$, is most useful for modifying the equations of motion for use in rotating non-inertial reference frames.

To proceed, we suppose for simplicity that the two origins coincide for all time, so that $O = \hat{O}$ and the reference frame $\mathcal{S}$ is thus rotating relative to the inertial frame, $\hat{\mathcal{S}}$.

We have two Cartesian bases, $\{\mathbf{e}_i\}$ and $\{\hat{\mathbf{e}}_i\}$, and we may expand the same vector $\mathbf{r}$ in both bases as

$$\mathbf{r} = \sum_{i=1}^{3} r_i\, \mathbf{e}_i = \sum_{i=1}^{3} \hat{r}_i\, \hat{\mathbf{e}}_i.$$

Here r_i are the *components of* $\mathbf{r}$ *in the reference frame* $\mathcal{S}$, while $\hat{r}_i$ are the components in the reference frame $\hat{\mathcal{S}}$.

The velocity of the particle is

$$\begin{aligned}\dot{\mathbf{r}} &= \sum_{i=1}^{3} \frac{\mathrm{d}\hat{r}_i}{\mathrm{d}t}\,\hat{\mathbf{e}}_i = \sum_{i=1}^{3}(\dot{r}_i\,\mathbf{e}_i + r_i\,\dot{\mathbf{e}}_i) \;=\; \sum_{i=1}^{3}\dot{r}_i\,\mathbf{e}_i + \sum_{i=1}^{3} r_i\,\boldsymbol{\omega}\wedge\mathbf{e}_i \\ &= \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\mathbf{r}\ ,\end{aligned} \tag{8.5}$$

where we have introduced:

Definition The *time derivative* of $\mathbf{r} = \mathbf{r}(t)$, as would be measured by an observer co-rotating with the frame $\mathcal{S}$, is

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} \equiv \dot{r}_1\,\mathbf{e}_1 + \dot{r}_2\,\mathbf{e}_2 + \dot{r}_3\,\mathbf{e}_3.$$

That is, we differentiate the *components* of $\mathbf{r}$ in the orthonormal basis $\{\mathbf{e}_i\}$ for $\mathcal{S}$.

One should always clarify the meaning of "$\dot{\mathbf{r}}$" when there are two general reference frames being used. We will *always* mean the time derivative in the *inertial reference frame* $\hat{\mathcal{S}}$ and hence we have shown:

Proposition (The Coriolis formula)

$$\dot{\mathbf{r}} \;=\; \sum_{i=1}^{3}\frac{\mathrm{d}\hat{r}_i}{\mathrm{d}t}\,\hat{\mathbf{e}}_i \equiv \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\hat{\mathcal{S}}} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\mathbf{r}\ , \tag{8.6}$$

where $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$.

For rigid body dynamics we will be interested in the velocity of points $\mathbf{r}$ that are *fixed relative to the rotating frame* $\mathcal{S}$. By definition this means that the first term on the right hand side of (8.6) is zero, and hence we may simply write

$$\dot{\mathbf{r}} \equiv \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\hat{\mathcal{S}}} \;=\; \boldsymbol{\omega}\wedge\mathbf{r}\ . \tag{8.7}$$

Definition In general we may write $\boldsymbol{\omega} = \omega\,\mathbf{n}$, where $\omega = \omega(t)$ is the *angular velocity*, and $\mathbf{n} = \mathbf{n}(t)$ is of unit magnitude and the *instantaneous axis of rotation*.

Note For $\omega > 0$, the direction of rotation is given by the right hand rule (point your right thumb in the direction of $\mathbf{n}$ and the direction of rotation is that of your right hand fingers, from proximal to distal).

In particular for cylindrical polars with $\boldsymbol{\omega} = \omega\mathbf{k}$, the rotation is about the direction $\mathbf{k}$ with $\dot{\theta} = \omega$.

Aside (Not lectured) For geometric intuition, consider the change $\delta\mathbf{r}$ in $\mathbf{r}$ in a small time interval δt, i.e. $\delta\mathbf{r} = \boldsymbol{\omega} \wedge \mathbf{r}\,\delta t$. As seen in the inertial frame $\hat{\mathcal{S}}$ and shown in Fig, (20), this is a rotation of $\mathbf{r}$ through an angle $|\boldsymbol{\omega}|\delta t$ about an axis parallel to the vector $\boldsymbol{\omega}$, with the direction as indicated. Hence $\mathbf{r}$ rotates around a cone with its vertex at the origin and symmetry axis $\boldsymbol{\omega}$, with the orientation shown and angular speed $|\boldsymbol{\omega}|$.

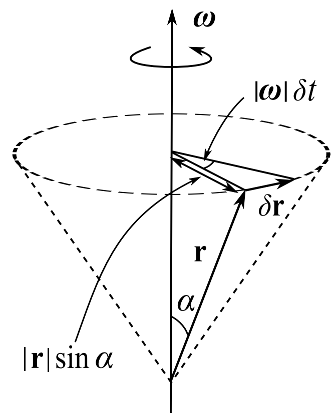


Figure 20: As seen in the inertial frame $\hat{\mathcal{S}}$, the position vector $\mathbf{r}$ of a point P fixed in the frame $\mathcal{S}$ changes by $\delta\mathbf{r} = \boldsymbol{\omega} \wedge \mathbf{r}\,\delta t$ in a small time interval δt. This is a rotation of $\mathbf{r}$ through an angle $|\boldsymbol{\omega}|\delta t$ about an axis parallel to the vector $\boldsymbol{\omega}$. The direction of rotation is as indicated and given by the right hand rule.

8.2 Rigid bodies

A *rigid body* may be defined as any distribution of mass for which the distance between any two points is fixed.

A simple model for this is to take a finite number of point particles, as in section 7.1, but with the constraint that the position vectors $\mathbf{r}_I$ $(I = 1, \ldots, N)$ satisfy

$$|\mathbf{r}_I - \mathbf{r}_J| = c_{IJ}, \text{ constant},$$

thus ensuring that the body retains its size, shape and distribution of mass.

One might imagine the $\mathbf{r}_I$ as the positions of atoms in a solid, with the constraints arising from inter-molecular forces. We assume these constraint forces satisfy the strong form of Newton's third law.

For now we will use this point particle model, but later we will model a rigid body as a continuous distribution of matter, which may be regarded as a limit of the point particle model in which the number of particles tends to infinity.

Choose a point O that is fixed in the body. For example, in the point particle model this could be one of the particles, although as we shall see below it will often be convenient to take this to be the *centre of mass*. We denote the position vector of O as $\mathbf{x} = \mathbf{x}(t)$, where this is measured from the origin $\hat{O}$ of the inertial frame $\hat{\mathcal{S}}$. We may then write

$$\mathbf{R}_I \;=\; \mathbf{x} + \mathbf{r}_I \;, \qquad I = 1, \ldots, N \;, \tag{8.8}$$

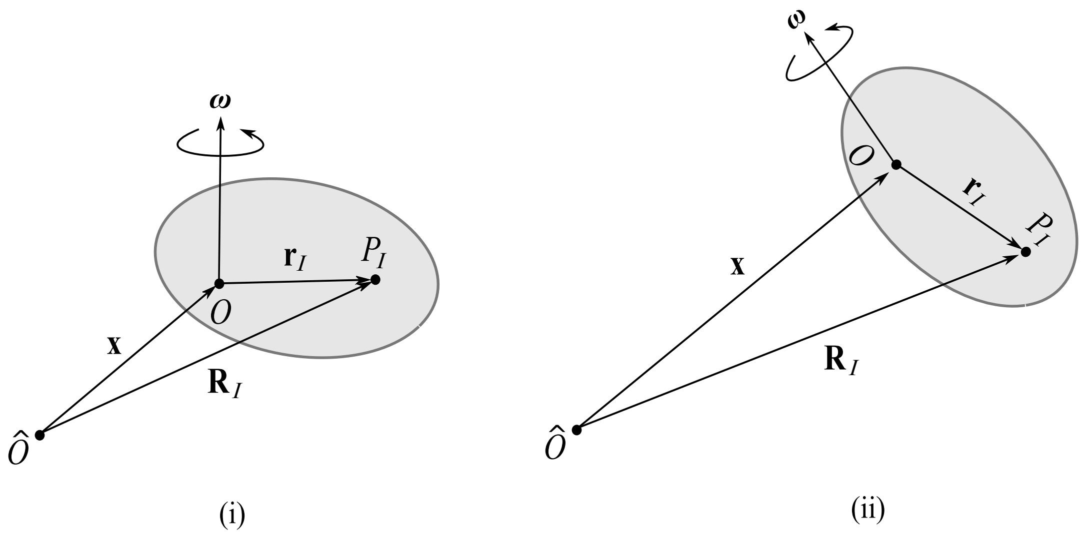


Figure 21: We fix a point O in the rigid body, which is taken to be the origin of the rest frame $\mathcal{S}$ of the body. The frame $\mathcal{S}$ has angular velocity $\boldsymbol{\omega}$, and its origin O has position vector $\mathbf{x}$ relative to the origin $\hat{O}$ of an inertial frame $\hat{\mathcal{S}}$. The body particles P_I have position vectors $\mathbf{r}_I$, measured from O. Figures (i) and (ii) show the same body at two different times.

so that $\mathbf{R}_I$ and $\mathbf{r}_I$ are the positions of the body particles, as measured from $\hat{O}$ and O, respectively. See Figure 21.

Definition The *rest frame* $\mathcal{S}$ of the rigid body is a reference frame, with origin O, with respect to which the $\mathbf{r}_I$ are fixed (at rest), *i.e.*

$$\left(\frac{\mathrm{d}\mathbf{r}_I}{\mathrm{d}t}\right)_{\mathcal{S}} = \mathbf{0} \quad \text{for all} \quad I = 1, \ldots, N.$$

Aside. The existence of such a frame is really equivalent to what we mean by a rigid body in the first place. Provided the matter distribution is not all along a line, the rest frame is defined uniquely by the body, up to a *constant* rotation of its axes and a translation of the origin by a *constant* vector (relative to $\mathcal{S}$).

Using Eqn. (8.7) we then have the important result that

$$\dot{\mathbf{R}}_I \;=\; \dot{\mathbf{x}} + \dot{\mathbf{r}}_I \;=\; \mathbf{v}_O + \boldsymbol{\omega} \wedge \mathbf{r}_I \; . \tag{8.9}$$

Here $\mathbf{v}_O = \dot{\mathbf{x}}$ is the velocity of O, as measured in the inertial frame $\hat{\mathcal{S}}$, while $\boldsymbol{\omega}$ is the angular velocity of the rest frame $\mathcal{S}$ with respect to $\hat{\mathcal{S}}$.

As we already mentioned, a natural choice for O is the centre of mass G of the body. This means that $\mathbf{x} = \mathbf{R}_G$, in the notation of section 7.1.

With $\mathbf{R}_I = \mathbf{R}_G + \mathbf{r}_I$ the position vectors of the particles relative to the inertial frame origin, so that $\mathbf{r}_I \to \mathbf{R}_I$ in the definition of the centre of mass, we have

$$\mathbf{R}_G \;=\; \frac{1}{M}\sum_{I=1}^{N} m_I \mathbf{R}_I \;=\; \frac{1}{M}\sum_{I=1}^{N} m_I(\mathbf{R}_G + \mathbf{r}_I) \;=\; \mathbf{R}_G + \frac{1}{M}\sum_{I=1}^{N} m_I \mathbf{r}_I \; .$$

Hence

$$\sum_{I=1}^{N} m_I \mathbf{r}_I \;=\; \mathbf{0}. \qquad (8.10)$$

We now re-examine the formulas for the total linear and angular momentum from section 7.1, and also look at the total kinetic energy.

We take the origin of the general reference to be the rigid body centre of mass, $O = G$, unless otherwise stated.

Linear momentum

We already know from Eqn (7.3) that $\mathbf{P} = M\dot{\mathbf{R}}_G = M\mathbf{v}_G$, but it is interesting to see this explicitly for a rigid body:

$$\mathbf{P} \;=\; \sum_{I=1}^{N} m_I \dot{\mathbf{R}}_I \;=\; \sum_{I=1}^{N} m_I (\dot{\mathbf{R}}_G + \boldsymbol{\omega} \wedge \mathbf{r}_I) \;=\; M\dot{\mathbf{R}}_G + \boldsymbol{\omega} \wedge \left(\sum_{I=1}^{N} m_I \mathbf{r}_I \right) \;=\; M\dot{\mathbf{R}}_G \; .$$

where the last equality uses Eqn (8.10).

In summary: the total momentum is the same as if all the mass, M, was at the centre of mass G.

Angular momentum

The total angular momentum **about the centre of mass** $O = G$ is by definition

$$\mathbf{L}_G \;=\; \sum_{I=1}^{N} \mathbf{r}_I \wedge m_I \dot{\mathbf{R}}_I \;=\; \sum_{I=1}^{N} m_I \mathbf{r}_I \wedge (\dot{\mathbf{R}}_G + \boldsymbol{\omega} \wedge \mathbf{r}_I) \;=\; \sum_{I=1}^{N} m_I \mathbf{r}_I \wedge (\boldsymbol{\omega} \wedge \mathbf{r}_I) \; .$$

Using the vector identity $\mathbf{r}_I \wedge (\boldsymbol{\omega} \wedge \mathbf{r}_I) = (\mathbf{r}_I \cdot \mathbf{r}_I)\boldsymbol{\omega} - (\mathbf{r}_I \cdot \boldsymbol{\omega})\mathbf{r}_I$, we may write

$$\mathbf{L}_G \;=\; \sum_{I=1}^{N} m_I \left[(\mathbf{r}_I \cdot \mathbf{r}_I)\boldsymbol{\omega} - (\mathbf{r}_I \cdot \boldsymbol{\omega})\mathbf{r}_I \right] \; . \qquad (8.11)$$

Definition The *inertia tensor* $\mathcal{I} = \mathcal{I}^{(O)} = (\mathcal{I}_{ij}^{(O)})$ of the rigid body, about a point O fixed in the body, is defined as

$$\mathcal{I}_{ij} \;=\; \sum_{I=1}^{N} m_I \left[(\mathbf{r}_I \cdot \mathbf{r}_I)\delta_{ij} - r_{I\,i}\, r_{I\,j} \right] \; . \qquad (8.12)$$

Here $\mathbf{r}_I = \sum_{i=1}^{3} r_{I\,i}\, \mathbf{e}_i$ are the position vectors of the body particles, in the rest frame basis $\{\mathbf{e}_i\}$.

Hence we may write the total angular momentum (8.11) in matrix notation as

$$\mathbf{L}_G \;=\; \mathcal{I}^{(G)} \boldsymbol{\omega} \;=\; \sum_{i,j=1}^{3} \mathcal{I}_{ij}^{(G)} \omega_j \, \mathbf{e}_i \; , \qquad (8.13)$$

where the coefficients of $\boldsymbol{\omega}$ are given via $\boldsymbol{\omega} = \sum_{k=1}^{3} \omega_k \mathbf{e}_k$.

Note that

- The label "tensor" is a common nomenclature in theoretical physics that here simply emphasises the *inertia tensor* maps a vector to a vector, in particular it maps angular velocity to angular momentum.
- The inertia tensor is defined in the rest frame of the body, and so is intrinsic to the body itself, and in particular independent of time t.
- The inertia depends on a choice of origin O, fixed in the body.
- The inertia tensor need not be proportional to the identity so that angular momentum and velocity are not parallel in general, leading to complex rigid body dynamics.
- The inertia tensor is symmetric, $\mathcal{I} = \mathcal{I}^{\mathrm{T}}$ (and real). By the Spectral Theorem in Linear Algebra II there is a change of basis by an orthogonal matrix $\mathcal{P}$ such that $\mathcal{P}\mathcal{I}\mathcal{P}^{\mathrm{T}}$ is diagonal, which simplifies the dynamics (somewhat).

Kinetic energy. The total kinetic energy of the body, as measured in the inertial frame, is

$$T = \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{R}}_I|^2 = \frac{1}{2} \sum_{I=1}^{N} m_I \left[|\dot{\mathbf{R}}_G|^2 + 2\dot{\mathbf{R}}_G \cdot (\boldsymbol{\omega} \wedge \mathbf{r}_I) + (\boldsymbol{\omega} \wedge \mathbf{r}_I) \cdot (\boldsymbol{\omega} \wedge \mathbf{r}_I) \right] .$$

The middle term on the right hand side is again zero, as $\sum_I m_I \mathbf{r}_I = \mathbf{0}$.

On the other hand we may rewrite the last term using a vector identity and the expression for the inertia tensor

$$\sum_{I=1}^{N} m_I (\boldsymbol{\omega} \wedge \mathbf{r}_I) \cdot (\boldsymbol{\omega} \wedge \mathbf{r}_I) = \sum_{I=1}^{N} m_I [\mathbf{r}_I^2 \boldsymbol{\omega}^2 - (\mathbf{r}_I \cdot \boldsymbol{\omega})^2] = \boldsymbol{\omega} \cdot \mathbf{L}_G.$$

to give

$$T = \frac{1}{2} M |\dot{\mathbf{R}}_G|^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}_G . \tag{8.14}$$

Hence the total kinetic energy is the sum of

(i) the kinetic energy for the centre of mass motion relative to $\hat{O}$

(ii) the *rotational kinetic energy* about G.

Definition The *rotational kinetic energy* about the centre of mass G is

$$T_{\mathrm{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}_G = \frac{1}{2} \boldsymbol{\omega}^{\mathrm{T}} \mathcal{I}^{(G)} \boldsymbol{\omega} = \frac{1}{2} \sum_{i,j=1}^{3} \mathcal{I}_{ij}^{(G)} \, \omega_i \, \omega_j . \tag{8.15}$$

Continuous mass distributions

For a *continuous* distribution of matter, rather than a point particle model, we assume the distribution of mass in the body is defined by a density $\rho(\mathbf{r})$.

Partition the volume in to a collection of small regions with volume $\delta x \, \delta y \, \delta z$ centred at

$$\mathbf{r} = \sum_{k=1}^{3} r_k \mathbf{e}_k = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3,$$

Linear motion	Angular (rotational) motion
Mass M	Inertia tensor $\mathcal{I}$ = "rotational mass"
Linear velocity $\dot{\mathbf{R}}$	Angular velocity $\boldsymbol{\omega}$
Linear speed $\lvert\dot{\mathbf{R}}\rvert$	Angular speed $\omega = \lvert\boldsymbol{\omega}\rvert$
Linear momentum $\mathbf{P} = M\dot{\mathbf{R}}$	Angular momentum $\mathbf{L} = \mathcal{I}\,\boldsymbol{\omega}$
Kinetic energy $\frac{1}{2}M\lvert\dot{\mathbf{R}}\rvert^2$	Rotational kinetic energy $\frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\mathcal{I}\,\boldsymbol{\omega}$
Equation of motion: $\dot{\mathbf{P}} = \mathbf{F}^{\mathrm{ext}}$	Angular equation of motion $\dot{\mathbf{L}} = \boldsymbol{\tau}^{\mathrm{ext}}$

Table 1: Contrasting linear motion with angular (rotational) motion. Each linear quantity has a corresponding angular counterpart. The inertia tensor may be viewed as a sort of "rotational mass". The equations of motion in the last line will be used in subsection 8.3 below.

with mass

$$\delta m = \rho(\mathbf{r})\,\delta x\,\delta y\,\delta z,$$

where $\mathbf{r}$ is measured from the reference fixed in the rigid body, O.

This effectively replaces

$$m_I \to \delta m = \rho(\mathbf{r})\delta V, \qquad \mathbf{r}_I \to \mathbf{r}$$

in the point particle model.

Summing over these regions and taking the limit that number of the individual integration regions becomes increasing large but their size increasing small, while always still partitioning the volume, the total mass becomes the volume integral

$$M \;=\; \iiint_{\mathrm{body}} \rho(\mathbf{r})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ . \tag{8.16}$$

Similarly, the inertia tensor (8.12) becomes

$$\mathcal{I}_{ij} \;=\; \iiint_{\mathrm{body}} \rho(\mathbf{r})\,[(\mathbf{r}\cdot\mathbf{r})\delta_{ij} - r_i r_j]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ . \tag{8.17}$$

Here $\mathbf{r} = (r_1, r_2, r_3) = (x, y, z)$, so that the last equation more explicitly reads

$$\mathcal{I} \;=\; \iiint_{\mathrm{body}} \rho(\mathbf{r}) \begin{pmatrix} y^2+z^2 & -xy & -zx \\ -xy & z^2+x^2 & -yz \\ -zx & -yz & x^2+y^2 \end{pmatrix} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ . \tag{8.18}$$

Definition The *moment of inertia about an axis parallel to the unit vector* $\mathbf{n}$ *through* O is $I = \mathbf{n}^{\mathrm{T}}\,\mathcal{I}\,\mathbf{n}$.

In particular, the diagonal entries in (8.18) are the *moments of inertia* about the three axes. The off-diagonal entries are called the *products of inertia*.

8.2.1 Examples

Example Determine the inertia tensor about the centre of mass for a uniform rectangular cuboid of mass M, and side lengths $2a$, $2b$, $2c$.

The density ρ is constant as the cuboid is uniform, and hence $\rho = M/(8abc)$.

The centre of mass is the origin of the cuboid by symmetry, and we take Cartesian axes aligned with the edges.

It is then straightforward to see that the products of inertia in this basis are zero; for example

$$\mathcal{I}_{12}^{(G)} \;=\; -\frac{M}{8abc}\int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c} xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \;=\; 0\;, \tag{8.19}$$

either by calculation or noting the integrand is of odd parity in x or y.

We next compute

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c} \rho\, x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \;=\; \frac{M}{8abc}\left[\frac{1}{3}x^3\right]_{-a}^{a} 2b\cdot 2c \;=\; \frac{Ma^2}{3}\;. \tag{8.20}$$

The integrals involving y^2 and z^2 are of course similar, and we deduce that

$$\mathcal{I}^{(G)} \;=\; \frac{M}{3}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & c^2+a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}\;. \tag{8.21}$$

■

The inertia tensor (8.21) is diagonal in this last example. One can always find a basis in which it is diagonal, as already mentioned (Spectral Theorem, Linear Algebra II).

Definition On writing $\mathcal{I}$ with respect to a basis where it is diagonal, so that

$$\mathcal{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}\,,$$

the eigenvalues I_i of $\mathcal{I}$, $i = 1,2,3$, are called the *principal moments of inertia*. The corresponding eigenvectors, with which the axes $\mathbf{e}_i$ are aligned once $\mathcal{I}$ is diagonal, are called the *principal axes*.

A rigid body thus in general determines its own natural choice of rest frame: the origin is the centre of mass G, while the axes are the principal axes. In this frame the inertia tensor about G is diagonal. This is the *natural* choice of rest frame, but it may not always be the most convenient choice.

We may also consider two-dimensional bodies, such as a thin flat disc, or one-dimensional bodies such as a rigid rod. In this case one replaces the density ρ by a *surface density*, or *line density*, respectively, and integrates over the surface or curve, respectively.

Two Dimensional Example. Determine the moment of inertia tensor about the centre of mass for a thin uniform disc with radius a and mass M.

The surface density is $\rho = M/(\pi a^2)$, and by symmetry the centre of mass must be at the centre of the disc.

Taking the centre of the disc to be the origin, with the disc lying in the (x, y) plane at $z = 0$, we may introduce polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ in this plane.

We then have

$$\mathcal{I}_{11}^{(G)} \;=\; \iint \rho\, y^2\, \mathrm{d}x\, \mathrm{d}y \;=\; \frac{M}{\pi a^2}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2 \sin^2\theta\, r\, \mathrm{d}r\, \mathrm{d}\theta \;=\; \frac{1}{4}Ma^2\,. \tag{8.22}$$

Notice here that the integrand is $\rho(y^2 + z^2) = \rho\, y^2$, as the body is two-dimensional and lies in the plane $z = 0$.

By symmetry we must have $\mathcal{I}_{11}^{(G)} = \mathcal{I}_{22}^{(G)}$. We also need

$$\mathcal{I}_{33}^{(G)} \;=\; \iint \rho\,(x^2 + y^2)\, \mathrm{d}x\, \mathrm{d}y \;= \mathcal{I}_{11}^{(G)} + \mathcal{I}_{22}^{(G)} =\; \frac{1}{2}Ma^2\,. \tag{8.23}$$

We have $\mathcal{I}_{13}^{(G)} = \mathcal{I}_{23}^{(G)} = 0$ as the disc lies in the plane $z = 0$. Also

$$\mathcal{I}_{12}^{(G)} \;=\; -\iint \rho\, xy\, \mathrm{d}x\, \mathrm{d}y \;= 0. \tag{8.24}$$

by explicit calculation or by antisymmetry on the mapping $x \to -x$ in the integrand.

Thus

$$\mathcal{I}^{(G)} \;=\; \frac{Ma^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\,. \tag{8.25}$$

■

One Dimensional Example. Determine the moment of inertia at one end of a uniform straight rod of length l, mass M for any axis perpendicular to the rod.

The line density of the rod is $\rho = M/l$.

We take $\mathbf{r} = (x, 0, 0)$, so that $x \in [0, l]$ parametrizes the distances of points in the rod from one end at $x = 0$. Then we consider $\boldsymbol{n} = (0, 1, 0)$ as one axis perpendicular to rod.

Then

$$\mathcal{I}_{22} = \int_{x=0}^{l} \rho\, x^2\, \mathrm{d}x \;=\; \frac{M}{l}\cdot\left[\frac{1}{3}x^3\right]_0^l \;=\; \frac{1}{3}Ml^2\,.$$

We also note choosing any other axis perpendicular to $(1, 0, 0)$ will generate the same results by symmetry, as may be explicitly checked. ■

8.3 Simple rigid body motion

In this section we study some simple examples of rigid body motion. In general the instantaneous axis of rotation – the direction that $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ points – can depend on time, e.g. throwing a chopping board into the air. The inertia tensor in this case is modelled by the uniform rectangular cuboid example.

Here we consider the simpler situation where the direction of the axis of rotation is fixed. Hence the rotation is then described purely by the angular velocity $\omega(t)$ with $\boldsymbol{\omega} = \omega(t)\boldsymbol{n}$, where the direction of rotation is the unit vector $\boldsymbol{n}$.

Equations of Motion The centre of mass G of the rigid body satisfies Newton's second law

$$M\ddot{\mathbf{R}}_G \;=\; \dot{\mathbf{P}} \;=\; \mathbf{F}^{\text{ext}} \,, \tag{8.26}$$

where $\mathbf{F}^{\text{ext}}$ is the total external force acting on the body (see Eqn. 7.6 for the derivation).

Its rotation can be determined via

$$\dot{\mathbf{L}}_G \;=\; \boldsymbol{\tau}_G^{\text{ext}} \,, \tag{8.27}$$

where $\boldsymbol{\tau}_G^{\text{ext}}$ is the total external torque about G (see Eqn. 7.11 for the derivation).

Example. Cylinder rolling down an inclined plane. Consider a uniform circular cylinder of length l, radius a and mass M. The cylinder rolls under gravity, *without slipping*, down a plane inclined at an angle φ to the horizontal, such that every circular cross section of the cylinder lies in a plane spanned by the gravitational acceleration, $\mathbf{g}$, and the line of greatest slope of the inclined plane, as pictured in Fig. 22. Determine the motion of the cylinder.

Solution: The motion is effectively two-dimensional by its symmetry, as shown Fig. 22, and hence we only need to consider the vertical plane through a line of greatest slope of the inclined plane and the centre of mass G of the cylinder.

The absence of slip for the cylinder entails that if x is the distance travelled down the slope and θ is the angle through which the cylinder has turned then, for all times,

$$x \;=\; a\,\theta \,. \tag{8.28}$$

The rotation is purely along the axis of symmetry of the cylinder, which points into the page in Figure 22, through G. Taking this to be the $\mathbf{e}_3$ direction of the cylinder body fixed reference frame $\mathcal{S}$, the angular velocity vector is

$$\boldsymbol{\omega} \;=\; (0, 0, \dot{\theta}) \,. \tag{8.29}$$

We next need the inertia tensor of the cylinder, about G. This is of the form (see Problem Sheet 7)

$$\mathcal{I}^{(G)} \;=\; \frac{1}{2}Ma^2 \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{8.30}$$

Thus the angular momentum of the cylinder about G is simply

$$\mathbf{L}_G \;=\; (0, 0, I_3\,\dot{\theta}) \,, \qquad \text{where} \qquad I_3 \;=\; \frac{1}{2}Ma^2 \,. \tag{8.31}$$

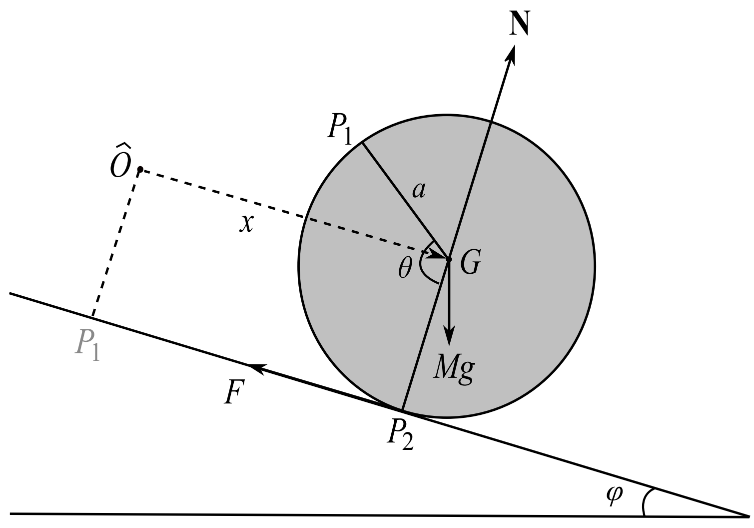


Figure 22: A cross-section through a circular cylinder rolling down a plane inclined at angle φ to the horizontal. The cylinder has radius a, and the distance travelled down the plane from a fixed origin $\hat{O}$ is x. The point of contact with the plane is labelled P_2, and a fixed point on the cylinder is labelled P_1. The angle between the radius vectors at P_1 and P_2 is θ, which is the angle through which the cylinder has rolled. A frictional force F acts at P_2 up the plane; a normal reaction $\mathbf{N}$ also acts at P_2. The gravitational force induces a total external force of $M\boldsymbol{g}$ on the cylinder, depicted by the arrow labelled Mg in the diagram.

The rotational form of Newton's second law, Eqn. (8.27), requires us to find the external torque $\boldsymbol{\tau}_G^{\rm ext}$ about G.

There are three forces acting: the normal reaction $\mathbf{N}$, gravity, and a frictional force $\mathbf{F}$ of magnitude $F = |\mathbf{F}|$ at the point of contact P_2 – see Figure 22. Physically, the friction force is required in order for the cylinder not to slip.

The Normal reaction force pass through G, and thus have zero moments about G. Gravity does not exert a torque about the Centre of Mass, as seen from Eqn. (7.12).

Thus the only contribution to the torque is from the friction force:

$$\boldsymbol{\tau}_G^{\rm ext} \;=\; \overrightarrow{GP_2} \wedge \mathbf{F} \;=\; a\,F\,\mathbf{e}_3 \; . \tag{8.32}$$

The sign here is easily determined using the right hand rule. Equation (8.27) thus gives

$$\dot{\mathbf{L}}_G \;=\; (0,0,I_3\,\ddot{\theta}) \;=\; \boldsymbol{\tau}_G^{\rm ext} \;=\; (0,0,a\,F) \qquad \Longrightarrow \qquad I_3\,\ddot{\theta} \;=\; a\,F \; . \tag{8.33}$$

Noting the centre of mass motion is in a straight line down the plane, without loss we have $\mathbf{R}_G(t) = x(t)\hat{\mathbf{e}}_1$ by choice of the Cartesian basis of the inertial reference frame $\hat{\mathcal{S}}$ in which the slope is at rest.

Then, Newton's second law for the linear momentum of the centre of mass (8.26), resolved in the $\hat{\mathbf{e}}_1$ direction, gives

$$M\ddot{x} \;=\; -F + Mg\sin\varphi \; . \tag{8.34}$$

Using $\theta = x/a$ and (8.33) we may eliminate F, θ to obtain giving

$$M\ddot{x} \;=\; -\frac{I_3}{a^2}\ddot{x} + Mg\sin\varphi \tag{8.35}$$

and hence the equation of motion

$$\ddot{x} \;=\; \frac{2}{3}g\sin\varphi \;, \tag{8.36}$$

and thus the motion is given by

$$x(t) = \frac{1}{3}g\sin\varphi\; t^2 + u_0 t + x_0,$$

where $u_0,\ x_0$ are constants. ∎

In contrast, for a point particle sliding down the inclined plane without friction. The equation of motion is $\ddot{x} = g\sin\varphi$ and thus the acceleration of the rolling cylinder is thus reduced by a factor of 2/3 compared to the point particle.

One can equivalently solve the last problem by thinking about energy. For this we need to know the gravitational potential energy of a rigid body:

Proposition The total gravitational potential energy of a rigid body in a uniform gravitational field is as if all the mass was located at the centre of mass G. That is, with $\boldsymbol{g} = -g\mathbf{k}$, the the gravitational potential energy is

$$V \;=\; Mg\,Z_G \;, \tag{8.37}$$

where Z_G is the z coordinate of the centre of mass G.

Proof: Thinking of the rigid body as made up of masses $\delta m = \rho(\mathbf{r})\,\delta x\,\delta y\,\delta z$ at positions $\mathbf{R} = \mathbf{R}_G + \mathbf{r} = (X, Y, Z)$ relative to the origin $\hat{O}$ of an inertial frame, these each have potential energy $\delta m\, g\, Z$. The total potential energy is hence

$$V \;=\; \iiint_{\text{body}} \rho(\mathbf{r})\, g\, Z\, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \;=\; Mg\,Z_G \;, \tag{8.38}$$

where the last step follows since by definition (7.2)

$$M\mathbf{R}_G \;=\; \iiint_{\text{body}} \rho(\mathbf{r})\,\mathbf{R}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \;, \tag{8.39}$$

and $\mathbf{R}_G = (X_G, Y_G, Z_G)$. ∎

Recall from (8.14) and (8.15) that the kinetic energy is

$$T \;=\; \frac{1}{2}M|\dot{\mathbf{R}}_G|^2 + \frac{1}{2}\sum_{i=1}^{3}\mathcal{I}_{ij}^{(G)}\,\omega_i\,\omega_j \;. \tag{8.40}$$

Example (Rolling cylinder again): The cylinder in our example rotates about a fixed axis $\mathbf{e}_3$ with angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_3$ and moment of inertia $I_{33} = I_3 = Ma^2/2$. Then, noting $x = a\theta$ by no-slip, (8.40) simplifies to

$$T = \frac{1}{2}M|\dot{\mathbf{R}}_G|^2 + \frac{1}{2}I_3\,\omega^2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{4}Ma^2\,\dot{\theta}^2 = \frac{3}{4}M\dot{x}^2. \tag{8.41}$$

From (8.38) the gravitational potential energy is

$$V = Mg\,Z_G = -Mg\,x\sin\varphi\ . \tag{8.42}$$

Thus the total energy is

$$E = T + V = \frac{3}{4}M\dot{x}^2 - Mg\,x\sin\varphi\ , \tag{8.43}$$

where we have substituted for θ in terms of x using (8.28).

Since there is a frictional force F acting one might be worried that this energy is not conserved. However, the point of contact P_2 does not slip relative to the stationary slope and thus instantaneously has the same speed as the slope, that is zero. Hence P_2 is always instantaneously at rest, which means that the friction does no work.

As usual the normal reaction also does no work, and so energy is indeed conserved.

Thus we find the equation of motion by setting the time derivative of (8.43) to zero, which yields:

$$0 = \dot{E} = \frac{3}{2}M\dot{x}\,\ddot{x} - Mg\,\dot{x}\sin\varphi = M\dot{x}\left[\frac{3}{2}\ddot{x} - g\,\sin\varphi\right]\ . \tag{8.44}$$

generating the equation of motion via the term in the square bracket. Note the degenerate case of $\dot{x} = 0$ at a point in time still entails $3\ddot{x}/2 - g\,\sin\varphi = 0$ by smoothness (but the case of $\dot{x} = 0$ for all time gives $\dot{\theta} = 0$ for all time and thus $0 = aF = F - mg\sin\varphi$ from the force and angular momentum balances, which has no solution for $g\sin\varphi > 0$; thus $\dot{x} = 0$ is not a physical solution). ■

One may instead reinterpret the above as independent confirmation that the friction force does no work given the equations of motion, i.e. $3\ddot{x}/2 - g\,\sin\varphi = 0$.

Example (Heavy pendulum): A *heavy pendulum* consists of a uniform rigid rod of mass M and length l, pivoted freely at one end at the origin O. The rod swings freely in a vertical plane under gravity. Determine the equation of motion for θ, the angle the rod makes with the vertical.

Solution: Notice in this example that we may take the origin $\hat{O}$ of the inertial frame to be the same point as the end of the rod O. It is then easier to consider the angular momentum about O, rather than about G.

We make use of polar coordinates in the plane of motion

$$\mathbf{e}_r = -\cos\theta\,\mathbf{k} + \sin\theta\,\mathbf{i}, \quad \mathbf{e}_\theta = \sin\theta\,\mathbf{k} + \cos\theta\,\mathbf{i},$$

where the vector $\mathbf{j}$ points into the page in Figure 23. The latter is the axis of rotation of the rod, so we may immediately write the angular velocity vector $\boldsymbol{\omega} = -\dot{\theta}\,\mathbf{j}$. Here the sign is easily checked using the right hand rule.

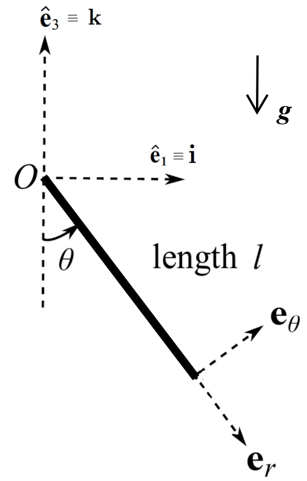


Figure 23: A heavy pendulum consisting of a uniform heavy rod of length l, mass M, depicted as the thick solid line. The polar coordinate system basis $\mathbf{e}_r$, $\mathbf{e}_\theta$ used in the main text is also shown.

In the one dimensional rod example of Section (8.2.1) we calculated the moment of inertia about the axis $\mathbf{j}$ through O for the heavy rod, giving $I = \frac{1}{3}Ml^2$. Thus the angular momentum in the $\mathbf{j}$ direction is

$$\mathbf{j} \cdot \mathbf{L}_O = -\dot{\theta}\,\mathbf{j} \cdot \mathcal{I}^{(O)}\mathbf{j} = -\frac{1}{3}Ml^2\dot{\theta}.$$

From Eqn (7.11) with $P = O = \hat{O}$, a fixed point in an inertial reference frame, we have $\dot{\mathbf{L}}_O = \boldsymbol{\tau}_O^{\text{ext}}$. The total external torque here just arises from the weight of the rod, and we may use Eqn (7.12) on noting the centre of mass G is halfway along the rod, by symmetry, to give:

$$\boldsymbol{\tau}_O^{\text{ext}} \;=\; -(\mathbf{R}_G - \mathbf{O}) \wedge (Mg\,\mathbf{k}) \;=\; -\frac{l}{2}\mathbf{e}_r \wedge Mg\,\mathbf{k} \;=\; \frac{1}{2}Mgl\sin\theta\,\mathbf{j}\;, \tag{8.45}$$

where in the last step we have used $\mathbf{e}_r \wedge \mathbf{k} = -\sin\theta\,\mathbf{j}$. Putting everything together, the angular equation of motion reads

$$\dot{\mathbf{L}}_O \;=\; -I\,\ddot{\theta}\,\mathbf{j} \;=\; \frac{1}{2}Mgl\sin\theta\,\mathbf{j} \;=\; \boldsymbol{\tau}_O^{\text{ext}}\;. \tag{8.46}$$

Using $I = \frac{1}{3}Ml^2$ hence gives the equation of motion

$$\ddot{\theta} \;=\; -\frac{3g}{2l}\sin\theta\;. \tag{8.47}$$

There is an extra factor of 3/2 compared with a simple pendulum of the same mass M and length l – see (5.10). In other words, a heavy pendulum behaves exactly the same as a *simple pendulum* with 2/3 of the length. ■

8.4 Newton's laws in a non-inertial frame

Throughout these lectures we have emphasized that Newton's laws, in particular the second law, should always be formulated in an inertial frame. By definition, this is a frame of reference in which

Newton's first law holds.

However, the Earth is rotating about its axis once per day. A fixed frame relative to the surface of the Earth is then only approximately an inertial frame. What effect does this have, and more generally can we formulate Newton's laws in a general reference frame?

We begin with the same framework as section 8.1: $\hat{\mathcal{S}}$ is a fixed inertial frame with origin $\hat{O}$, and $\mathcal{S}$ is another frame whose origin O is at position vector $\mathbf{x}(t)$, measured from $\hat{O}$. See Figure 24.

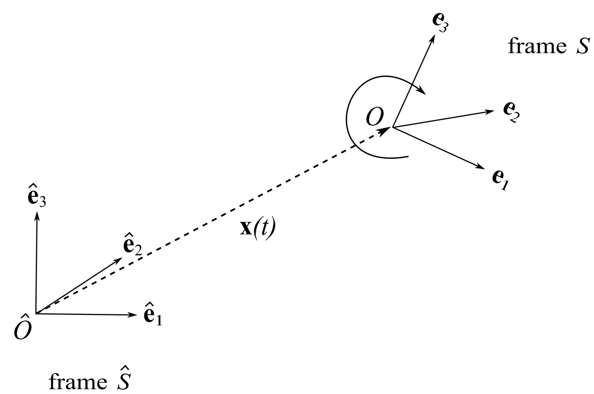


Figure 24: The reference frame $\hat{\mathcal{S}}$ is a fixed inertial reference frame, and Newton's laws of motion are written in this reference frame. With respect to $\hat{\mathcal{S}}$, a general reference frame $\mathcal{S}$ has origin O at position vector $\mathbf{x} = \mathbf{x}(t)$ as measured from $\hat{O}$, and its coordinate axes may be rotating, so that $\mathbf{e}_i = \mathbf{e}_i(t)$.

Suppose that a point particle has position vector $\mathbf{R}$ measured from $\hat{O}$, and $\mathbf{r}$ measured from O, as in (8.8). Then

$$\mathbf{R} \;=\; \mathbf{x} + \mathbf{r}\ . \tag{8.48}$$

Recall also from section 8.1 that

Definition The *time derivative* of a vector $\mathbf{q} = \mathbf{q}(t)$ in a frame $\mathcal{S}$ is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}} \mathbf{q} \;=\; \sum_{i=1}^{3} \dot{q}_i\, \mathbf{e}_i\ , \tag{8.49}$$

where $\mathbf{q} = \sum_{i=1}^{3} q_i\, \mathbf{e}_i$ and $\{\mathbf{e}_i\}$ is the orthonormal basis for $\mathcal{S}$. That is, we differentiate the *components* of $\mathbf{q}$ in this basis, with respect to time t.

The Coriolis formula (8.6) relates the time derivatives of the same vector $\mathbf{q}$ in $\mathcal{S}$ and $\hat{\mathcal{S}}$ as

$$\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{\hat{\mathcal{S}}} \;=\; \left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega} \wedge \mathbf{q}\ , \tag{8.50}$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ is the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$.

By definition then the accelerations $\hat{\mathbf{a}}$ and $\mathbf{a}$ of our particle, as measured in the frames $\hat{\mathcal{S}}$ and $\mathcal{S}$, respectively, are

$$\begin{aligned}\hat{\mathbf{a}} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\hat{\mathcal{S}}}\mathbf{R} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\hat{\mathcal{S}}}(\mathbf{x}+\mathbf{r}) = \left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right)_{\hat{\mathcal{S}}} + \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{\hat{\mathcal{S}}} ,\\ \mathbf{a} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\mathcal{S}}\mathbf{r} .\end{aligned} \tag{8.51}$$

In order to write down Newton's second law in the frame $\mathcal{S}$ we need the following result:

Proposition The accelerations in the two frames are related by

$$\hat{\mathbf{a}} = \mathbf{a} + \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}} \wedge \mathbf{r} + 2\boldsymbol{\omega} \wedge \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) + \mathbf{A} , \tag{8.52}$$

where we have defined $\mathbf{A} = \left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right)_{\hat{\mathcal{S}}}$, which is the acceleration of O relative to $\hat{\mathcal{S}}$.

Proof: We have, with use of the Coriolis formula (8.50),

$$\begin{aligned}\hat{\mathbf{a}} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\hat{\mathcal{S}}}(\mathbf{x}+\mathbf{r}) = \mathbf{A} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\hat{\mathcal{S}}}\mathbf{r}\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\hat{\mathcal{S}}}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\right]\mathbf{r} + \mathbf{A} = \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\right]\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\right]\mathbf{r} + \mathbf{A}\\ &= \mathbf{a} + \boldsymbol{\omega} \wedge \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}}(\boldsymbol{\omega}\wedge\mathbf{r}) + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) + \mathbf{A}\\ &= \mathbf{a} + \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}} \wedge \mathbf{r} + 2\boldsymbol{\omega} \wedge \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) + \mathbf{A} .\end{aligned} \tag{8.53}$$

■

Notice that using the Coriolis formula (8.50) we have

$$\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\hat{\mathcal{S}}} = \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\boldsymbol{\omega} = \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}} , \tag{8.54}$$

so that the time derivative of $\boldsymbol{\omega}$ is the same in either reference frame.

Newton's second law for a particle of mass m in the inertial frame $\hat{\mathcal{S}}$ is

$$m\hat{\mathbf{a}} = \mathbf{F} , \tag{8.55}$$

where $\mathbf{F}$ is the external force acting. Substituting for $\hat{\mathbf{a}}$ in terms of $\mathbf{a}$ using (8.52) in Newton's second law, we thus have:

Theorem With $\mathbf{r}$ the particle's position measured from the origin O of $\mathcal{S}$ Newton's second law in the frame $\mathcal{S}$ is

$$m\mathbf{a} = \mathbf{F} - m\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}} \wedge \mathbf{r} - 2m\,\boldsymbol{\omega} \wedge \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} - m\,\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) - m\mathbf{A} . \tag{8.56}$$

where, relative to the inertial reference frame $\hat{\mathcal{S}}$, $\mathbf{A}$ is the acceleration of O and $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{S}$.

The additional terms on the right hand side of (8.56) may be interpreted as "fictitious forces":

$$\begin{aligned} \mathbf{F}_1 &= -m\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}} \wedge \mathbf{r} \ , \qquad & \mathbf{F}_2 &= -2m\,\boldsymbol{\omega} \wedge \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} \ , \\ \mathbf{F}_3 &= -m\,\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) \ , \qquad & \mathbf{F}_4 &= -m\mathbf{A} \ . \end{aligned} \tag{8.57}$$

These may be regarded as corrections to the force in $\mathbf{F} = m\mathbf{a}$ due to the fact that the frame $\mathcal{S}$ is accelerating.

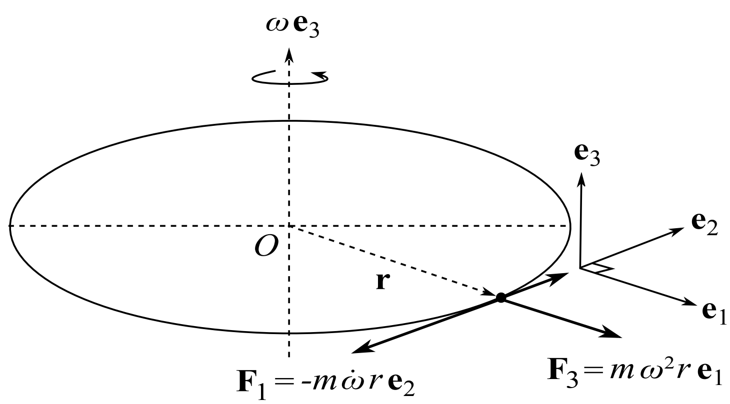


Figure 25: The Euler force $\mathbf{F}_1$ and centrifugal force $\mathbf{F}_3$ in a roundabout frame. Here $O = \hat{O}$, $\mathbf{e}_1$ is a unit vector directed radially outwards, $\mathbf{e}_2$ is a unit vector orthogonal to this in the horizontal plane of the roundabout, and $\mathbf{e}_3$ is a unit vector in the direction of the axis of rotation. The position vector of particle of mass m is $\mathbf{R} = \mathbf{r} = r\,\mathbf{e}_1$. The Euler force is then $\mathbf{F}_1 = -m\dot{\omega}\,\mathbf{e}_3 \wedge \mathbf{r} = -m\dot{\omega}r\,\mathbf{e}_2$ while the centrifugal force is $-m\omega\,\mathbf{e}_3 \wedge (\omega\,\mathbf{e}_3 \wedge \mathbf{r}) = m\omega^2 r\,\mathbf{e}_1$.

1. The force $\mathbf{F}_1$ is the *Euler force*, and arises from the *angular acceleration* of $\mathcal{S}$; in particular it is zero for a frame rotating at constant angular velocity. See Fig. 25.
2. The force $\mathbf{F}_2$ is the *Coriolis force*, and depends on the velocity

$$\mathbf{v} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}}$$

   of the particle as measured in $\mathcal{S}$.
3. The force $\mathbf{F}_3$ is the *centrifugal force*. It lies in a plane through $\mathbf{r}$ and $\boldsymbol{\omega}$, is perpendicular to the axis of rotation $\boldsymbol{\omega}$, and is directed away from the axis. This is the force you experience standing on a roundabout, that seems to throw you outwards; see Fig. 25.
4. Finally, $\mathbf{F}_4$ is simply due to the acceleration of the origin O. For example, this force cancels the Earth's gravitational field in a freely falling frame.

Corollary The frame $\mathcal{S}$ is inertial if and only if $\mathbf{A} = \mathbf{0} = \boldsymbol{\omega}$. That is, the origin O is not accelerating, and the basis is not rotating.

Proof (not lectured): First note that the frame $\mathcal{S}$ being inertial means that any particle with no force acting ($\mathbf{F} = \mathbf{0}$) moves at constant velocity in the frame $\mathcal{S}$. If $\mathbf{A} = \mathbf{0} = \boldsymbol{\omega}$ then (8.56) with $\mathbf{F} = \mathbf{0}$ immediately gives $\mathbf{a} = \mathbf{0}$, and hence the particle moves with constant velocity in $\mathcal{S}$. Conversely, suppose that $\mathbf{F} = \mathbf{0}$ and a particle moves with constant velocity $\mathbf{r}(t) = \mathbf{u}\,t + \mathbf{r}_0$ in $\mathcal{S}$. Here $\mathbf{u}$ and $\mathbf{r}_0$ are *arbitrary* constant vectors in $\mathcal{S}$ (effectively integration constants from integrating $\mathbf{a} = \mathbf{0}$). First setting $\mathbf{u} = \mathbf{r}_0 = \mathbf{0}$ (so the particle is fixed at the origin of $\mathcal{S}$), we immediately deduce from substituting $\mathbf{r} \equiv \mathbf{0}$ into (8.56) that $\mathbf{A} = \mathbf{0}$. Next, for *fixed* time $t = t_0$ we may set $\mathbf{r}_0 = -\mathbf{u}\,t_0$ (so the particle is at the origin of $\mathcal{S}$ at time t_0), and again substitute for $\mathbf{r}(t) = \mathbf{u}\,t + \mathbf{r}_0$ into (8.56). Evaluated at time $t = t_0$, the only term that survives is the Coriolis term $-2m\,\boldsymbol{\omega}(t_0) \wedge \mathbf{u}$, which must be zero for all $\mathbf{u}$. But this implies that $\boldsymbol{\omega}(t_0) = \mathbf{0}$, and since t_0 was arbitrary hence $\boldsymbol{\omega} \equiv \mathbf{0}$. ■

8.4.1 Examples

In the two examples that follow the origin O of the rotating frame $\mathcal{S}$ may be taken to coincide with $\hat{O}$, so that $\mathbf{x} = \mathbf{0}$ and the position vectors in the two frames are equal $\mathbf{R} = \mathbf{r}$.

Example *Bead on a rotating, smooth, straight horizontal wire.* Consider a bead (point particle), of mass m, sliding on a frictionless straight horizontal wire that is fixed at $O = \hat{O}$, and rotating in the horizontal plane at constant angular velocity ω.

Show that the centrifugal force is in the direction of the wire away from O and determine the motion of the bead.

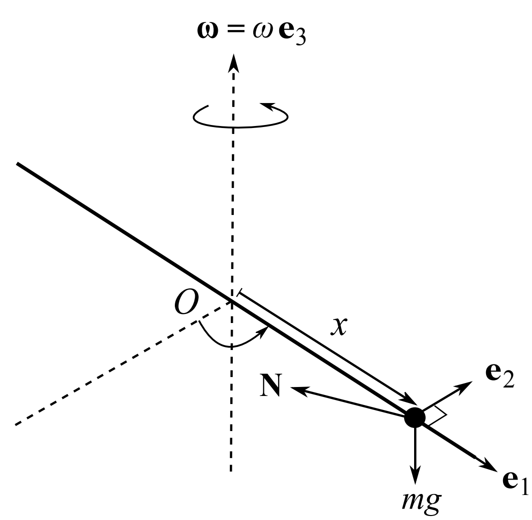


Figure 26: The bead on the rotating horizontal wire. The forces acting on the bead are $-mg\,\mathbf{k}$ and the normal reaction $\mathbf{N}$ perpendicular to the wire.

Solution:

We choose a right-handed Cartesian basis $\{\mathbf{e}_i\}$ for the wire-fixed frame, $\mathcal{S}$, with

- $\mathbf{e}_1$ is a unit vector pointing along the wire
- $\mathbf{e}_2$ is a unit horizontal vector normal to the wire in the plane of rotation
- $\mathbf{e}_3$ is a unit vector vertically.

- Note that $\mathbf{e}_3 = \mathbf{k}$ and without loss we take this to be $\hat{\mathbf{e}}_3$ of the inertial frame.

The position of the bead is $\mathbf{r} = \mathbf{R} = x(t)\,\mathbf{e}_1$, while the angular velocity of the frame is $\boldsymbol{\omega} = \omega\,\mathbf{e}_3$. With $\mathbf{N}$ the normal reaction of the wire on the bead, the total force on the bead for use of Newton's second law in an inertial reference frame is

$$\mathbf{F} \;=\; \mathbf{N} - mg\,\mathbf{e}_3\ . \tag{8.58}$$

However, as we are working in a frame that is rotating, so we must use Eqn. (8.56). Since $\boldsymbol{\omega}$ is constant and $\mathbf{A} = \mathbf{0}$ we have

$$m\mathbf{a} = \mathbf{F} - 2m\,\boldsymbol{\omega} \wedge \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} - m\,\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})\ . \tag{8.59}$$

Noting

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} = \dot{x}\mathbf{e}_1$$

by the definition of the time derivative in the $\mathcal{S}$ frame, the Coriolis force is

$$\mathbf{F}_2 \;=\; -2m\,\boldsymbol{\omega} \wedge \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} \;=\; -2m\omega\,\mathbf{e}_3 \wedge \dot{x}\,\mathbf{e}_1 \;=\; -2m\omega\,\dot{x}\,\mathbf{e}_2\ . \tag{8.60}$$

The Centrifugal force is

$$\mathbf{F}_3 \;=\; -m\,\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) \;=\; -m\omega^2\,\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge x\,\mathbf{e}_1) \;=\; m\omega^2\,x\,\mathbf{e}_1\ . \tag{8.61}$$

and we see that this acts to push the bead out along the wire away from O.

As in section 5.1, the wire being *smooth* means that the normal reaction $\mathbf{N}$ has no component along the wire, $\mathbf{N} \cdot \mathbf{e}_1 = 0$. Thus taking the dot product of (8.69) with $\mathbf{e}_1$ gives

$$m\ddot{x} \;=\; m\omega^2\,x\ , \tag{8.62}$$

with general solution

$$x(t) \;= A\,\mathrm{e}^{\omega t} + B\,\mathrm{e}^{-\omega t}\ . \tag{8.63}$$

For example, if the bead starts at a distance $x = a$ from O with $\dot{x} = 0$ at time $t = 0$, then

$$x(t) \;=\; \frac{a}{2}(\mathrm{e}^{\omega t} + \mathrm{e}^{-\omega t}) \;=\; a\,\cosh\omega t\ , \tag{8.64}$$

and the bead flings outwards on the wire, with $x(t)$ growing exponentially with t. ■

Example *Bead on a rotating smooth hoop.* A circular hoop of radius a rotates at constant angular velocity ω about a vertical diameter. A bead slides smoothy on the hoop and has a position vector which makes an angle φ with the vertical, as in Fig. 27. Show that the equation of motion is

$$\ddot{\varphi} + \left(\frac{g}{a} - \omega^2\cos\varphi\right)\sin\varphi \;=\; 0\ . \tag{8.65}$$

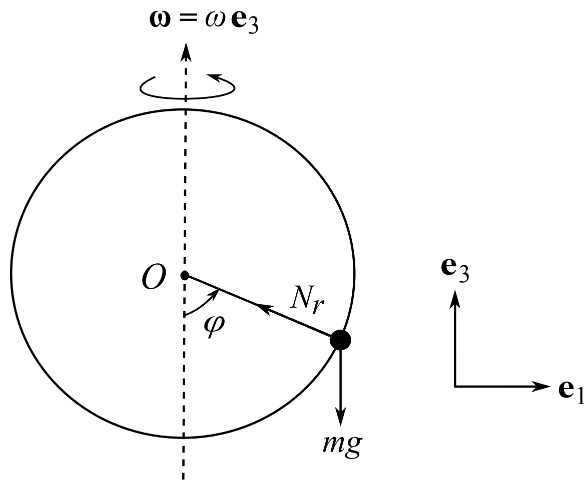


Figure 27: The bead on the rotating hoop of radius a. Here the figure shows the hoop at the instant at which it passes through the plane of the page. The radial component, N_r, of the normal reaction $\mathbf{N}$ of the hoop on the bead is shown, while the N_2 component points into the page at this instant, which is the $\mathbf{e}_2$ direction.

Solution: We take the origins $O = \hat{O}$ to be the centre of the hoop, and the frame $\mathcal{S}$ to be the hoop-fixed frame. In particular we take $\mathbf{e}_1$ to be a horizontal unit vector and $\mathbf{e}_3$ to be a vertical unit vector, which define the (rotating) plane of the hoop.

Note as previously $\mathbf{e}_3 = \mathbf{k}$ and without loss we take this to be $\hat{\mathbf{e}}_3$ of the inertial frame.

The angular velocity is $\boldsymbol{\omega} = \omega\,\mathbf{e}_3$ and we may then parametrize the position of the bead as

$$\mathbf{r} \;=\; \mathbf{R} \;=\; a\,\sin\varphi\,\mathbf{e}_1 - a\,\cos\varphi\,\mathbf{e}_3\;. \tag{8.66}$$

We then compute the velocity and acceleration of the bead with respect to the rotating, hoop-fixed, frame:

$$\begin{aligned} \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} &= a\,\dot{\varphi}\,\cos\varphi\,\mathbf{e}_1 + a\,\dot{\varphi}\,\sin\varphi\,\mathbf{e}_3\;, \\ \mathbf{a} \;=\; \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{\mathcal{S}} &= a(\ddot{\varphi}\,\cos\varphi - \dot{\varphi}^2\sin\varphi)\,\mathbf{e}_1 + a(\ddot{\varphi}\,\sin\varphi + \dot{\varphi}^2\cos\varphi)\,\mathbf{e}_3\;. \end{aligned} \tag{8.67}$$

With $\mathbf{N}$ the normal reaction of the wire on the bead, the total force on the bead for use of Newton's second law in an inertial reference frame is

$$\mathbf{F} \;=\; \mathbf{N} - mg\,\mathbf{e}_3\;. \tag{8.68}$$

However, as we are working in a frame that is rotating, so we must use Eqn. (8.56). Since $\boldsymbol{\omega}$ is constant and $\mathbf{A} = \mathbf{0}$, since $O = \hat{O}$, we once more have

$$m\mathbf{a} = \mathbf{F} - 2m\,\boldsymbol{\omega}\wedge\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} - m\,\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r})\;. \tag{8.69}$$

Hence

$$\begin{aligned} m\mathbf{a} &= ma(\ddot{\varphi}\,\cos\varphi - \dot{\varphi}^2 \sin\varphi)\,\mathbf{e}_1 + ma(\ddot{\varphi}\,\sin\varphi + \dot{\varphi}^2\cos\varphi)\,\mathbf{e}_3 \\ &= \mathbf{F} - 2m\omega\,\mathbf{e}_3 \wedge (a\,\dot{\varphi}\,\cos\varphi\,\mathbf{e}_1 + a\,\dot{\varphi}\,\sin\varphi\,\mathbf{e}_3) \\ &\qquad\qquad - m\omega\,\mathbf{e}_3 \wedge [\omega\,\mathbf{e}_3 \wedge (a\,\sin\varphi\,\mathbf{e}_1 - a\,\cos\varphi\,\mathbf{e}_3)] \\ &= \mathbf{F} - 2m\omega\,\mathbf{e}_3 \wedge a\,\dot{\varphi}\,\cos\varphi\,\mathbf{e}_1 - m\omega\,\mathbf{e}_3 \wedge [\omega\,\mathbf{e}_3 \wedge a\,\sin\varphi\,\mathbf{e}_1] \\ &= \mathbf{F} - 2m\omega a\dot{\varphi}\,\cos\varphi\,\mathbf{e}_3 \wedge \mathbf{e}_1 - m\omega^2 a\sin\varphi\;\mathbf{e}_3 \wedge [\,\mathbf{e}_3 \wedge\,\mathbf{e}_1] \\ &= \mathbf{N} - mg\,\mathbf{e}_3 - 2m\omega\,a\,\dot{\varphi}\,\cos\varphi\,\mathbf{e}_2 + m\omega^2\,a\,\sin\varphi\,\mathbf{e}_1\ . \end{aligned} \tag{8.70}$$

where in the latter lines the vector products have been simplified and determined, and Eqn. (8.68) has been used.

The normal reaction $\mathbf{N}$ has a radial component N_r (see Figure 27) and a component N_2 into the page. We need to eliminate $\mathbf{N}$, as $\mathbf{N}$ does not feature in the equation of motion. Noting $\mathbf{N}$ is perpendicular to the tangent of the circular hoop,

$$\mathbf{t} \;=\; \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_3\ . \tag{8.71}$$

we take the dot product of (8.70) with $\mathbf{t}$.

This gives

$$ma\,\ddot{\varphi} \;=\; -mg\,\sin\varphi + m\omega^2\,a\,\sin\varphi\,\cos\varphi\ . \tag{8.72}$$

Dividing through by ma then gives the required equation of motion. ■